Some Illustrative Examples of Permutability of Fuzzy Operators and Fuzzy Relations

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Abstract

Composition of fuzzy operators often appears and it is natural to ask when the order of composition does not change the result. In previous papers, we characterized permutability in the case of fuzzy consequence operators and fuzzy interior operators. We also showed the connection between the permutability of the fuzzy relations and the permutability of their induced fuzzy operators. In this work we present some examples of permutability and non permutability of fuzzy operators and fuzzy relations in order to illustrate these results.

Keywords: Permutability, Fuzzy Consequence Operator, Fuzzy Preorder, Similarity Relation.

1 INTRODUCTION

Composition of fuzzy operators often appears in fields like fuzzy mathematical morphology or approximate reasoning. In fuzzy mathematical morphology, fuzzy operators are used as morphological filters for image processing [6], [7]. In approximate reasoning, fuzzy consequence operators perform the role of deriving consequences from certain premises and relations [5], [9], [11]. These two fields are closely related and several results can be transfered from one field to the other [10]. In previous papers [3], [4] we studied permutability of the composition of fuzzy consequence operators and fuzzy interior operators in a general context and we related it to the preservation of the operator type through composition. We also connected permutability of fuzzy relations with permutability of the operators they induce through Zadeh's compositional rule. We focused in the case of fuzzy consequence operators induced by fuzzy preorders and fuzzy indistinguishability relations. The aim of this paper is to show that all the studied cases exist and to provide examples to illustrate each of them.

Our paper is organized as follows: In Section 2 we recall the main definitions and results that will be used throughout the paper. In Section 3 we recall the main results about permutability from our previous work. Finally, in Section 4 we provide a collection of examples to illustrate each of the studied cases.

2 PRELIMINARIES

In this paper, X will denote a non-empty classical universal set, $[0,1]^X$ will be the set of all fuzzy subsets of X with truth values in $[0,1]$ and $*$ a left-continuous t-norm.

Definition 2.1. *A fuzzy (binary) relation on X is a map $R : X \times X \longrightarrow [0,1]$. Γ' will denote the set of fuzzy binary relations defined on X. A fuzzy relation $R \in \Gamma'$ is said to be:*

1. *Reflexive if $R(x,x) = 1 \quad \forall x \in X$*
2. *Symmetric if $R(x,y) = R(y,x) \quad \forall x,y \in X$*
3. *$*$-Transitive if $R(x,y) * R(y,z) \leq R(x,z) \quad \forall x,y,z \in X$*

A reflexive and $*$-transitive fuzzy relation is called a fuzzy $*$-preorder. *If it also satisfies symmetry, then it is called a* fuzzy $*$-similarity *or* $*$-indistinguishability relation. *Given $R,S \in \Gamma'$, we say that $R \leq S$ if and only if $R(x,y) \leq S(x,y)$ for all $x,y \in X$.*

Composition of fuzzy relations is given by the sup-$*$ composition.

Definition 2.2. *Let $R,S \in \Gamma'$ be fuzzy relations on a set X and $*$ a t-norm. The* ***sup-$*$ composition*** *of R and S is the fuzzy relation defined for all $x,y \in X$ by*

$$R \circ S(x,y) = \sup_{w \in X} \{R(x,w) * S(w,y)\} \tag{1}$$

The transitive closure of a fuzzy relation R is the smallest upper approximation of R which is $*$-transitive [2]. More precisely,

Definition 2.3. *We define the* $*$***-transitive closure*** $\overline{R}$ *of a fuzzy relation* R *as the fuzzy relation given by*

$$\overline{R} = \inf_{\substack{S\in\widehat{\Gamma}\\ R\leq S}} \{S\} \tag{2}$$

where $\widehat{\Gamma}$ *denotes the set of all* $*$*-transitive fuzzy relations in* X.

The explicit formula for the transitive closure is given by $\overline{R} = \sup_{n\in\mathbb{N}} R^n$ where the power of R is defined using the sup-$*$ composition. The $*$-transitive closure of a reflexive fuzzy relation is a fuzzy $*$-preorder and the $*$-transitive closure of a reflexive and symmetric relation is an $*$-indistinguishability relation.

A fuzzy operator is a map $C : [0,1]^X \longrightarrow [0,1]^X$. We denote Ω' the set of fuzzy operators defined from $[0,1]^X$ to $[0,1]^X$.

Definition 2.4. *A fuzzy operator* $C\in\Omega'$ *is called a* fuzzy consequence operator *(FCO for short) when it satisfies for all* $\mu,\nu\in[0,1]^X$*:*

1. *Inclusion* $\mu\subseteq C(\mu)$
2. *Monotony* $\mu\subseteq\nu\Rightarrow C(\mu)\subseteq C(\nu)$
3. *Idempotence* $C(C(\mu)) = C(\mu)$

Ω *will denote the set of all FCO on* X.

Definition 2.5. *[1] A fuzzy operator* $C\in\Omega'$ *is called a* fuzzy interior operator *(FIO for short) when it satisfies for all* $\mu,\nu\in[0,1]^X$*:*

1. *Anti-inclusion* $C(\mu)\subseteq\mu$
2. *Monotonicity* $\mu\subseteq\nu\Rightarrow C(\mu)\subseteq C(\nu)$
3. *Idempotence* $C(C(\mu)) = C(\mu)$

Λ *will denote the set of all FIO on* X.

The inclusion of fuzzy subsets is given by the pointwise order, i.e. $\mu\subseteq\nu$ if and only if $\mu(x)\leq\nu(x)$ for all $x\in X$. Given two fuzzy operators C_1, C_2 we say that $C_1 \leq C_2$ if $C_1(\mu) \subseteq C_2(\mu)$ for all $\mu\in[0,1]^X$.

Every fuzzy relation induces a fuzzy operator using Zadeh's compositional rule.

Definition 2.6. *Let* $R\in\Gamma'$ *be a fuzzy relation on* X*. The fuzzy operator induced by* R *through Zadeh's compositional rule is defined by*

$$C_R^*(\mu)(x) = \sup_{w\in X}\{\mu(w)*R(w,x)\} \tag{3}$$

Let us recall the definitions of fuzzy closure and fuzzy interior of a fuzzy operator C. The fuzzy closure is the smallest FCO which is greater than or equal to C. The fuzzy interior is the greatest FIO which is smaller than or equal to C. Formally,

Definition 2.7. *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be a fuzzy operator. We define the **fuzzy closure*** $\overline{C}$ ***of*** C *as the fuzzy operator given by*

$$\overline{C} = \inf_{\substack{\phi\in\Omega\\ C\leq\phi}}\{\phi\}\ . \tag{4}$$

Definition 2.8. *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be a fuzzy operator. We define the **fuzzy interior*** $\overset{\circ}{C}$ ***of*** C *as the fuzzy operator given by*

$$\overset{\circ}{C} = \sup_{\substack{\phi\in\Lambda\\ C\geq\phi}}\{\phi\}\ . \tag{5}$$

3 PERMUTABILITY

First of all, let us recall the definitions of permutability for fuzzy relations and fuzzy operators.

Definition 3.1. *Let* $R,S\in\Gamma'$ *be fuzzy relations. We say that* R *and* S ***permute*** *(or that* R *and* S *are permutable) if* $R\circ S = S\circ R$ *where* $\circ$ *is the sup-*$*$ *composition.*

Definition 3.2. *Let* C,C' *be fuzzy operators. We say that* C *and* C' ***permute*** *(or that* C *and* C' *are permutable) if* $C\circ C' = C'\circ C$ *where* $\circ$ *is the usual composition.*

3.1 PERMUTABILITY OF FCO AND FIO

In [3] we proved the following results which characterize permutability between FCO and the dual case of FIO. Proofs are provided there.

For two fuzzy consequence operators to permute it is necessary and sufficient that their composition gives a FCO in both directions:

Theorem 3.1. *Let* C,C' *be fuzzy consequence operators. Then,* C *and* C' *permute if and only if* $C\circ C'$ *and* $C'\circ C$ *are fuzzy consequence operators.*

Hence, permutability appears when the operator type is preserved.

Proposition 3.2. *Let* C,C' *be fuzzy consequence operators. Then,* $C\circ C'$ *is a fuzzy consequence operator if and only if* $C\circ C' = \max(C,C')$.

As a consequence of the previous result we have that C and C' permute if and only if both $C\circ C'$ and $C'\circ C$ coincide with $\max(C,C')$.

The dual results hold for fuzzy interior operators. For two fuzzy interior operators to permute it is necessary and sufficient that their composition gives a FIO in both directions. In this case, their composition is the interior of their minimum.

Theorem 3.3. *Let* C,C' *be fuzzy interior operators. Then,* C *and* C' *permute if and only if* $C\circ C'$ *and* $C'\circ C$ *are fuzzy interior operators.*

Proposition 3.4. *Let C, C' be fuzzy interior operators. Then, $C \circ C'$ is a fuzzy interior operator if and only if $C \circ C' = \overset{\circ}{\min}(C, C')$.*

Hence, C and C' permute if and only if both $C \circ C'$ and $C' \circ C$ coincide with $\overset{\circ}{\min}(C, C')$.

3.2 PERMUTABILITY OF FCO INDUCED BY FUZZY RELATIONS

It was proved in [13] that two $*$-indistinguishability relations defined on a finite set X permute if and only if $E \circ F$ is an $*$-indistinguishability relation. In this case, $E \circ F = \overline{\max(E, F)}$.

In [4], this result was extended to general fuzzy preorders and any set X, finite or not.

Theorem 3.5. *Let R and P be two fuzzy $*$-preorders on X. Then, R and P are permutable if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders. In this case, $R \circ P$ coincides with the $*$-transitive closure $\overline{\max(R, P)}$ of $\max(R, P)$.*

Notice that both compositions are needed in order to obtain permutability. In Section 4 we will illustrate this fact with an example.

Since indistinguishability relations are preorders that also satisfy the symmetric property, we can soften this constraint.

Proposition 3.6. *Let E and F be two $*$-indistinguishability relations on X. Then, E and F are permutable if and only if $E \circ F$ is a $*$-indistinguishability relation. In this case, $E \circ F$ coincides with the $*$-transitive closure $\overline{\max(E, F)}$ of $\max(E, F)$.*

Composition of fuzzy operators induced by fuzzy relations using Zadeh's compositional rule can be described in terms of the inducing relations as shown in the following proposition. This description makes natural to think that permutability of fuzzy relations is connected to permutability of the operators they induce.

Proposition 3.7. *Let R, S be two fuzzy relations and let C_R^* and C_S^* be the corresponding fuzzy operators induced through Zadeh's compositional rule. Then,*

$$C_R^* \circ C_S^* = C_{S \circ R}^* \tag{6}$$

where $S \circ R$ denotes the sup-$*$ composition of fuzzy relations.

Let us focus on the case of operators induced by fuzzy preorders. It is well known that fuzzy operators induced from fuzzy relations through Zadeh's compositional rule are fuzzy consequence operators if and only if the relation is a fuzzy preorder [8]. However, not every FCO can be induced by a preorder. The relation between permutability of fuzzy preorders and permutability of their induced consequence operators can be summarized in the following theorem.

Theorem 3.8. *[4] Let R, P be fuzzy $*$-preorders. Then,*

$$C_R^* \circ C_P^* = C_P^* \circ C_R^* \quad \Leftrightarrow \quad R \circ P = P \circ R$$

If the preorders are fuzzy indistinguishability relations, the induced operators behave specially well.

Proposition 3.9. *[12] Let E be a fuzzy $*$-indistinguishability relation and let C_E^* be the fuzzy operator induced through Zadeh's compositional rule. Then,*

1. *C_E^* is a fuzzy consequence operator.*
2. *$C_E^*(\bigcup_{i \in I} \mu_i) = \bigcup_{i \in I} C_E^*(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *$C_E^*(\{x\})(y) = C_E^*(\{y\})(x)$ for all $x, y \in X$ where $\{x\}$ denotes the singleton of x.*
4. *$C_E^*(\alpha * \mu) = \alpha * C_E^*(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

Moreover, every fuzzy operator satisfying conditions of Proposition 3.9 can be written in the form C_E^* for a certain $*$-indistinguishability relation E. Hence, there is a bijection between the set of $*$-indistinguishability relations and the set of fuzzy operators satisfying the conditions of Proposition 3.9.

Even if C_E^* and C_F^* do not permute, their composition always satisfy the following properties.

Proposition 3.10. *Let E, F be fuzzy $*$-indistinguishability relations. Then, $C_{E \circ F}^*$ satisfies properties 2, 4 of Proposition 3.9. Moreover, it satisfies the inclusion and monotony properties from the definition of FCO.*

In [4] we proved that permutability of operators induced by similarity relations can be characterized as follows:

Theorem 3.11. *Let E, F be $*$-indistinguishability relations. Then, their consequences C_E^* and C_F^* permute if and only if $E \circ F$ is an indistinguishability relation.*

Corollary 3.12. *Let C, C' be fuzzy operators satisfying all the conditions of Proposition 3.9. Then, C and C' permute if and only if $C \circ C'$ also satisfies all these conditions.*

4 EXAMPLES

The aim of this section is to illustrate the results given in Section 3 with different examples. We will provide cases of fuzzy operators and fuzzy relations that do and do not permute.

The first example shows that the condition in Theorem 3.5 cannot be softened. Preservation of the operator type in one direction is not enough to obtain permutability between fuzzy consequence operators.

Example 4.1. *Let X be a non empty classical set and let $\alpha, \beta \in \mathbb{R}$ such that $0 < \beta < \alpha < 1$. Let C' and C be FCO defined as follows:*

$$C'(\mu)(x) = \begin{cases} 1 & if \quad \mu(x) > \beta \\ \beta & if \quad \mu(x) \leq \beta \end{cases}$$

$$C(\mu)(x) = \begin{cases} 1 & if \quad \mu(x) > \alpha \\ \alpha & if \quad \mu(x) \leq \alpha \,. \end{cases}$$

Note that $C' \circ C$ is a FCO. In fact, $C' \circ C = \overline{\max(C, C')} = X$ where $X(\mu)(x) = 1$ for all $x \in X$ and $\mu \in [0,1]^X$. However, C and C' ***do not permute****. Indeed, one has*

$$(C \circ C')(\mu)(x) \begin{cases} 1 & if \quad \mu(x) > \beta \\ \alpha & if \quad \mu(x) \leq \beta \end{cases}$$

which is not a FCO since it does not satisfy idempotence.

The next example proves that there exist fuzzy consequence operators that permute.

Example 4.2. *Let X be a non empty classical set. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$ and let C' and C be defined as*

$$C'(\mu)(x) = \begin{cases} 1 & if \ x = x_1 \\ \mu(x) & otherwise \end{cases}$$

$$C(\mu)(x) = \begin{cases} 1 & if \ x = x_2 \\ \mu(x) & otherwise \end{cases}$$

Notice that C and C' are FCO and ***permute****, i.e. $C \circ C' = C' \circ C$, and by Thm. 3.2, their composition*

$$C' \circ C(\mu)(x) = \begin{cases} 1 & if \ \ x = x_1 \ \ or \ \ x = x_2 \\ \mu(x) & otherwise \end{cases}$$

is a FCO.

Let us show some cases of permutability and non-permutability of min-preorders. From Theorem 3.8, their induced consequence operators behave in the same way as the relations do.

For simplicity, we shall write C^*_R instead of $C^{\min}_R$.

Example 4.3. *Let R and P be fuzzy min-preorders (but not similarities) defined as follows:*

$$R = \begin{pmatrix} 1 & 0.3 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0.7 & 0.8 \\ 0.55 & 1 & 0.7 \\ 0.4 & 0.4 & 1 \end{pmatrix}$$

R and P ***permute*** *and therefore, their consequences also do.*

$$\overline{\max(R,P)} = R \circ P = P \circ R = \begin{pmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.4 & 1 \end{pmatrix}$$

Taking for example, $\mu = \begin{pmatrix} 0.2 & 0.8 & 0.5 \end{pmatrix}$, it is easy to see that

$$C^*_R \circ C^*_P(\mu) = C^*_P \circ C^*_R(\mu) = \begin{pmatrix} 0.7 & 0.8 & 0.75 \end{pmatrix}$$

Example 4.4. *Let Q and S be fuzzy min-preorders (but not similarities) defined as follows.*

$$Q = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.6 & 1 & 0.5 \\ 0.3 & 0.3 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0.3 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix}$$

Their compositions are given by

$$Q \circ S = \begin{pmatrix} 1 & 0.4 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix}$$

$$S \circ Q = \begin{pmatrix} 1 & 0.4 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.4 & 1 \end{pmatrix}$$

Notice that $\overline{\max(S,Q)} = S \circ Q$ but Q and S ***do not permute****. Taking $\mu = \begin{pmatrix} 0.2 & 0.3 & 0.5 \end{pmatrix}$, it is easy to see that*

$$C^*_Q \circ C^*_S(\mu) = \begin{pmatrix} 0.4 & 0.4 & 0.5 \end{pmatrix}$$
$$C^*_S \circ C^*_Q(\mu) = \begin{pmatrix} 0.4 & 0.3 & 0.5 \end{pmatrix}$$

Example 4.5. *Let E and F be fuzzy min-indistinguishability relations defined as follows.*

$$E = \begin{pmatrix} 1 & 0.8 & 0.7 & 0.7 \\ 0.8 & 1 & 0.7 & 0.8 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.8 & 0.7 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0.6 & 0.5 & 0.8 \\ 0.6 & 1 & 0.5 & 0.6 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.8 & 0.6 & 0.5 & 1 \end{pmatrix}$$

Notice that E and F ***permute****.*

$$\overline{\max(E,F)} = E \circ F = F \circ E = \begin{pmatrix} 1 & 0.8 & 0.7 & 0.8 \\ 0.8 & 1 & 0.7 & 0.8 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.8 & 0.8 & 0.7 & 1 \end{pmatrix}$$

Therefore, C_E^* *and* C_F^* *also* ***permute****.*

Example 4.6. *Let E and F be fuzzy min-indistinguishability relations defined as follows.*

$$E = \begin{pmatrix} 1 & 0.4 & 0.4 & 0.4 \\ 0.4 & 1 & 0.8 & 0.7 \\ 0.4 & 0.8 & 1 & 0.7 \\ 0.4 & 0.7 & 0.7 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0.5 & 0.7 & 0.8 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.7 & 0.5 & 1 & 0.7 \\ 0.8 & 0.5 & 0.7 & 1 \end{pmatrix}$$

E and F ***do not permute****.*

$$F \circ E = \begin{pmatrix} 1 & 0.7 & 0.7 & 0.8 \\ 0.5 & 1 & 0.8 & 0.7 \\ 0.7 & 0.8 & 1 & 0.7 \\ 0.8 & 0.7 & 0.7 & 1 \end{pmatrix}$$

$$E \circ F = \begin{pmatrix} 1 & 0.5 & 0.7 & 0.8 \\ 0.7 & 1 & 0.8 & 0.7 \\ 0.7 & 0.8 & 1 & 0.7 \\ 0.8 & 0.7 & 0.7 & 1 \end{pmatrix}$$

However, as seen in Proposition 3.10, both compositions $C_E^* \circ C_F^*$ *and* $C_F^* \circ C_E^*$ *satisfy properties 2, 4 of Proposition 3.9 and the inclusion and monotony properties from the definition of FCO.*

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