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# Extensions of Topological Abelian Groups

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# Extensions of Topological Abelian Groups

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This dissertation has been written under our supervision in the Doctoral Program in Complex Systems, and we approve its submission to the Defense Committee.

Signed on June 9, 2016

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*To my mother.*

**Abstract**

An extension of topological abelian groups is a short exact sequence $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ where all homomorphisms are assumed to be continuous and open onto their images. $E$ is said to split if it is equivalent to the trivial extension $0 \to H \to H \times G \to G \to 0$. It is known that $E$ splits if and only if $\imath(H)$ splits as a topological subgroup of $X$ (i.e. there exists a closed subgroup $L \leq X$ such that the map $\imath(H) \times L \to X;\ (\imath(h), l) \mapsto \imath(h) + l$ is a topological isomorphism).

Given $G$ and $H$, the set of all extensions of topological abelian groups of the form $0 \to H \to X \to G \to 0$, endowed with the Baer Sum is an abelian group denoted by $\mathrm{Ext}(G, H)$. This group will be trivial when every extension of the previous form splits.

This thesis is devoted to the following two important problems:

(a) Study the properties of Ext in the category of topological abelian groups. In particular find conditions under which $\mathrm{Ext}(G, H) = 0$.

(b) Find ways to describe the extensions of topological abelian groups of the form $0 \to H \to X \to G \to 0$ using the properties of $G$ and $H$.

Regarding (a),

- *We construct the group* $\mathrm{Ext}(G, H)$ *in the realm of topological abelian groups, and we obtain results concerning the behavior of this group when we take dense subgroups, open subgroups, products, coproducts and quotients in $G$ or $H$ (§3.2 - §3.5). As an example, we show that there exists a metrizable group topology $\tau$ on $\mathbb{R}$ such that* $\mathrm{Ext}((\mathbb{R}, \tau), \mathbb{R})$ *is an infinite dimensional vector space (7.2.11).*

- *We prove that* $\mathrm{Ext}(G, H) = 0$ *whenever $G$ is a product of locally precompact abelian groups and $H$ is a product of copies of $\mathbb{R}$ and $\mathbb{T}$ (7.1.9).*

Two notions will be the key to study (b): quasi-homomorphisms and continuous cross-sections.

A map $q : G \to H$ is called a *quasi-homomorphism* if $q(0) = 0$ and $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is continuous at 0. The quasi-homomorphism $q$ is said to be *approximable* if there exists a homomorphism $a : G \to H$ such that $q - a$ is continuous at 0. We will study the connections between quasi-homomorphisms and extensions. It turns out that approximable quasi-homomorphisms produce trivial extensions.

Given an extension of topological abelian groups $0 \to H \to X \xrightarrow{\pi} G \to 0$, a continuous map $s : G \to X$ is called a continuous *cross-section* of $E$ if $\pi \circ s = \mathrm{Id}_G$. If $E$ admits a continuous cross-section then $X$ is homeomorphic to $H \times G$.

We prove:

- *If $G$ is a $k_\omega$ zero-dimensional abelian group and $H$ is a compact abelian group, then every extension of the form $0 \to H \to X \to G \to 0$ admits a continuous cross-section (5.1.14).*
- *Let $H$ be either a Banach space or the unit circle $\mathbb{T}$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of topological abelian groups such that every quasi-homomorphism of the form $G_\alpha \to H$ is approximable. Then every quasi-homomorphism of the form $\prod_{\alpha<\kappa} G_\alpha \to H$ is approximable (6.4.4).*

**Resumen**

Una extensión de grupos topológicos abelianos es una sucesión exacta corta $E : 0 \to H \to X \to G \to 0$ en la cual los homomorfismos son continuos y abiertos sobre sus imágenes. Se dice que $E$ escinde cuando es equivalente a la extensión trivial $0 \to H \to H \times G \to G \to 0$. Es sabido que $E$ escinde si solo si $\imath(H)$ escinde como subgrupo topológico de $X$ (i.e. existe un subgrupo cerrado $L \leq X$ tal que la función $\imath(H) \times L \to X;\ (\imath(h), l) \mapsto \imath(h) + l$ es un isomorfismo topológico).

Dados $G$ y $H$, el conjunto de todas las extensiones de grupos topológicos abelianos de la forma $0 \to H \to X \to G \to 0$, dotado con la suma de Baer es un grupo abeliano denotado por $\mathrm{Ext}(G, H)$. Este grupo es trivial cuando toda extensión de la forma anterior escinde.

Esta tesis se centra en el estudio de los siguientes problemas:

(a) Estudiar las propiedades de grupo Ext en la categoría de grupos topológicos abelianos. En particular, encontrar condiciones en las que $\mathrm{Ext}(G, H) = 0$.

(b) Encontrar maneras de describir las extensiones de la forma $0 \to H \to X \to G \to 0$ usando las propiedades de $G$ y $H$.

Con respecto a (a),

- *Construimos el grupo* $\mathrm{Ext}(G, H)$ *en el contexto de grupos topológicos abelianos y obtenemos resultados que conciernen al comportamiento de este grupo cuando tomamos subgrupos densos, subgrupos abiertos, productos, coproductos y cocientes en $G$ o $H$. A modo de ejemplo, demostramos que existe $\tau$ una topología de grupo metrizable en $\mathbb{R}$ tal que* $\mathrm{Ext}((\mathbb{R}, \tau), \mathbb{R})$ *es un espacio vectorial de dimensión infinita. (7.2.11).*

- *Demostramos que* $\mathrm{Ext}(G, H) = 0$ *siempre $G$ sea un producto de grupos abelianos localmente precompactos y $H$ sea un producto de copias de $\mathbb{R}$ y $\mathbb{T}$ (7.1.9).*

Dos nociones serán fundamentales para el estudio de (b):
cuasi-homomorfismos y cross-sections continuas.

Una se dice que una función $q : G \to H$ es un *cuasi-homomorfismo* si $q(0) = 0$ y $(x, y) \mapsto q(x + y) - q(x) - q(y)$ es continuo en 0. El quasi-homomorfismo $q$ es aproximable si existe un homomorfismo $a : G \to H$ tal que $q - a$ es continuo en 0. Estudiaremos las conexiones entre cuasi-homomorfismos y extensiones. Los cuasi-homomorfismos aproximables producen extensiones triviales.

Dada una extensión de grupos topológicos $0 \to H \to X \xrightarrow{\pi} G \to 0$, una función continua $s : G \to H$ se dice que es una *cross-section* continua para $E$ si $\pi \circ s = \mathrm{Id}_G$. Si $E$ admite una cross-section continua, entonces $X$ es homeomorfo a $H \times G$.

Demostraremos que

- *Si* $G$ *es un grupo topológico abeliano* $k_\omega$ *y zero-dimensional y* $H$ *es un grupo abeliano compacto, entonces toda extensión de la forma* $0 \to H \to X \stackrel{\pi}{\to} G \to 0$ *admite una cross-section continua (5.1.14).*

- *Sea* $M$ *un espacio de Banach* $B$ *o el círculo unidad* $\mathbb{T}$*. Sea* $\{G_\alpha : \alpha < \kappa\}$ *una familia de grupos abelianos topológicos tales que para todo* $\alpha < \kappa$*, todo cuasi-homomorfismo de* $G_\alpha$ *en* $M$ *es aproximable. Entonces todo cuasi-homomorfismo de* $\prod_{\alpha<\kappa} G_\alpha$ *en* $M$ *es aproximable (6.4.4).*

# Introduction

An *extension of topological abelian groups* is a short exact sequence $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ where all homomorphisms are assumed to be continuous and open onto their images. The extension $E$ *splits* if $\imath(H)$ splits as a topological subgroup of $X$.

The study of the extensions of topological abelian groups started in 1951 (as far as the author knows) with the work of Calabi [Cal51]. In his dissertation Calabi adapts the notion of extension of abelian groups to the realm of topological groups [1] and he studies what is known as the *extension problem* in the context of topological groups:

**Problem 1.** *Given $G$ and $H$, study the properties of the extensions of topological abelian groups of the form $0 \to H \to X \to G \to 0$.*

Inspired by Calabi, Sze-tsen Hu published [Hu52], where he studies Problem 1 and investigates the relations between the different cohomology theories for topological groups established in the literature. Among other results, he describes the structure of all extensions that admit continuous cross-sections ([Hu52, 5.3, 5.4 p. 17]) and proves that if $A(X)$ is the free abelian topological group over a Tychonov space $X$ and an extension $0 \to H \to X \to A(X) \to 0$ admits a continuous cross-section (i.e. a continuous left inverse) then it splits [Hu52, 5.6].

In 1967 Moskowitz developed in his paper [Mos67] the homological algebra of the class $\mathcal{L}$ of locally compact abelian groups. He studied the extensions of locally compact abelian groups and proved that the only groups $H \in \mathcal{L}$ such that every extension of locally compact abelian groups of the form $0 \to H \to X \to G \to 0$ splits are those topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^\kappa$ for some $n < \omega$ and $\kappa$ an arbitrary ordinal. In [Mos67, Section VI] he introduces the group $\mathrm{Ext}(G, H)$ of all extensions of locally compact abelian groups of the form $0 \to H \to X \to G \to 0$ using the techniques developed for abelian groups by Mac Lane in [ML95]. Nevertheless, he is only interested in the case in which $H$ is elementary [2]. He uses the Ext

[1] In his work, Calabi prefers the definition that avoids the use of short exact sequences: given topological groups $G, X$ and $H$, $X$ is said to be an *extension of $G$ by $H$*, if it contains $H$ as a closed subgroup and $X/H$ is topologically isomorphic to $G$.

[2] $H$ is elementary if it is topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^\alpha \times \mathbb{Z}^m \times F$ where $n, m \in \omega$,

group to find several Hom-Ext exact sequences for various subclasses of $\mathcal{L}$ (see [Mos67, Th. 6.2]).

The study of homological constructions in $\mathcal{L}$ was continued by Fulp and Griffith in [FG71a] and [FG71b]. They studied the group $\mathrm{Ext}(G, H)$ for arbitrary groups $G$ and $H$ in $\mathcal{L}$, and found the following Hom-Ext sequences for the class $\mathcal{L}$: *Given an extension $0 \to A \to B \to C \to 0$ of topological abelian groups in $\mathcal{L}$, for every $G \in \mathcal{L}$ there are exact sequences of abelian groups*

$$\begin{aligned} 0 &\to \mathrm{CHom}(C, G) \to \mathrm{CHom}(B, G) \to \mathrm{CHom}(A, G) \\ &\to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0 \end{aligned}$$

$$\begin{aligned} 0 &\to \mathrm{CHom}(G, A) \to \mathrm{CHom}(G, B) \to \mathrm{CHom}(G, C) \\ &\to \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0 \end{aligned}$$

This result sharpened the ones developed by Moskowitz in that direction and provided many applications. These two authors started the study of the so called *splitting problem* on topological groups:

**Problem 2.** *Find conditions on two topological abelian groups $G, H$ that force $\mathrm{Ext}(G, H)$ to be trivial (that is, every extension of topological abelian groups of the form $0 \to H \to X \to G \to 0$ splits).*

They focused in the case in which $G$ and $H$ are in $\mathcal{L}$ (see [FG71a, Th. 5.1, Th. 5.2] and [FG71b, Th. 3.1, Th. 3.3, Cor. 3.5]). One year after the appearance of these two important papers, Fulp published [Ful72], obtaining more results in the direction of Problem 2. It is worth mentioning that Sahleh and Alijani in [SA14a] and [SA14b] recently continued the study of the group Ext in the class $\mathcal{L}$ and Problem 2 using the techniques of Fulp and Griffith.

An extension of topological vector spaces is a short exact sequence $0 \to Y \to X \to Z \to 0$ in which $Y, X, Z$ are topological vector spaces and the maps are relatively open and continuous linear mappings. It is clear that an extension of topological vector spaces is in particular an extension of topological abelian groups.

In 1984 Kalton, Peck and Roberts published the seminal work [KPR84] in which they undertook an extensive study of the extensions of topological vector spaces. In particular, they were interested in the following problem:

**Problem 3.** *Which topological vector spaces $Z$ have the property that every extension of topological vector spaces $0 \to \mathbb{R} \to X \to Z \to 0$ splits?*

The spaces that satisfy such property are called $\mathcal{K}$-spaces. These three authors proved that the Banach spaces $\ell^p$ and $L^p$ are $\mathcal{K}$-spaces for $0 < p < 1$

$\alpha$ is any ordinal number and $F$ is finite (see [Mos67, p. 394]) (see [KPR84, Th. 5.7, Cor. 5.16]) and for $1 < p < \infty$ (see [KPR84, Th. 5.18]). On the other hand, Kalton ([Kal78]) Ribe ([Rib79]) and Roberts ([Rob77]) proved independently that $\ell^1$ is not a $\mathcal{K}$-space. Problem 3 was later studied by Domański in his paper [Dom85].

The key that the authors use in [KPR84] and [Dom85] to study the behavior of the extensions of topological vector spaces is the notion of *quasi-linear mapping*. A map $q : Z \to Y$ between topological vector spaces is *quasi-linear* if it satisfies the following properties:

(a) $q(0) = 0$.

(b) The map $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is continuous at the origin.

(c) The map $(\lambda, x) \mapsto q(\lambda x) - \lambda q(x)$ is continuous at the origin.

(This is the definition given by Domański in [Dom85, §3] which differs from the one in [KPR84, page 85]). It turns out that for every quasi-linear mapping $q : Z \to Y$ there is a canonical procedure to construct an extension of topological vector spaces $E_q : 0 \to Y \to Y \oplus_q Z \to Z \to 0$. Furthermore $E_q$ is trivial (i.e. *splits*) if and only if $q$ is approximable (in the sense that there exists a linear mapping $a : Z \to Y$ which does not need to be continuous but satisfies that $q - a$ is continuous at the origin).

In his paper [Cab03], Cabello noticed that these constructions could be translated very naturally to the context of topological abelian groups. Given topological abelian groups $G, H$, he defined a *quasi-homomorphism* as a map $q : G \to H$ satisfying the conditions (a) and (b) above. In [Cab03, Lemmas 2 and 3] Cabello points out that, as it is the case with quasi-linear mappings, using a quasi-homomorphism $q : G \to H$ one can produce an extension of topological abelian groups $E_q : H \to H \oplus_q G \to G \to 0$ which will be trivial if and only if $q$ is approximable (in the sense that there is a homomorphism $a : G \to H$ with $q - a$ continuous at the origin).

In 1980, Cattaneo explored the connections between the theory of extensions of topological abelian groups and that of extensions of topological vector spaces. He was interested in the following problem:

**Problem 4.** *If $Y, Z$ are topological vector spaces and $E : 0 \to Y \to X \to Z \to 0$ is an extension of topological abelian groups, what additional properties do we need to impose on $Y$ and $Z$ so that $X$ admits a compatible topological vector space structure that transforms $E$ into an extension of topological vector spaces?*

He proved in [Cat80, Prop. 2] that such a topological vector space structure can be constructed on $X$ if $Y$ is Fréchet and $Z$ is metrizable and complete. In a latter work Cabello showed that the same is also true if both $Y$ and $Z$ are complete and locally bounded ([Cab04, Th. 4]).

The aim of this dissertation is to investigate the extensions of topological abelian groups in a general setting (outside $\mathcal{L}$) and to continue the work of Fulp and Griffith in the study of problems 1 and 2. As a mean to do this, we will explore the applications of the notions of quasi-homomorphism and cross-section in the framework of topological abelian groups. Parallelly, we will study the extensions of topological vector spaces and Problem 4 using the techniques and ideas of Kalton, Peck, Ribe, Roberts, Domański, Cattaneo and Cabello.

This document is organized as follows:

Chapter 2 provides a detailed explanation of the theory of extensions of topological abelian groups (which will be assumed to be Hausdorff). We will start chapter 3 with the introduction of the Ext group in the realm of topological abelian groups, then in §3.2 to §3.5 we will study the behavior of this group when we take dense subgroups, open subgroups, products, coproducts and quotients. We will show that *for every topological abelian groups $G$ and $H$*

- *If $H$ is Čech-complete,* $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(\varrho G, H)$*, where $\varrho G$ is the Raĭkov completion of $G$, (3.2.4).*
- *If $H$ is divisible and $A$ is an open subgroup of $G$,* $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(A, H)$ *(3.3.2).*
- $\mathrm{Ext}(G, \prod_{\alpha<\kappa} H_\alpha) \cong \prod_{\alpha<\kappa} \mathrm{Ext}(G, H_\alpha)$ *for every family $\{H_\alpha : \alpha < \kappa\}$ of topological abelian groups (3.4.1).*
- $\mathrm{Ext}(\bigoplus_{\alpha<\omega} G_\alpha, H) \cong \prod_{\alpha<\omega} \mathrm{Ext}(G_\alpha, H)$ *where $\bigoplus_{\alpha<\omega} G_\alpha$ is a countable coproduct of topological abelian groups (3.4.4).*

We will also apply these properties to attack Problem 2 and show that several results proven by Fulp, Griffith, Sahleh and Alijani in [FG71a],[FG71b], [SA14a], and [SA14b] can be formulated in a more general context.

Chapter 4 deals with the extensions of topological vector spaces. §4.1 is concerned with constructing the Ext group in the class of topological vector spaces and studying its properties using the techniques of chapter 3. In the second section of this chapter we will tackle Problem 4 by proving that *given an extension of topological abelian groups* $0 \to Y \to X \to Z \to 0$*, $X$ is a topological vector space in any of the following situations:*

- *$Z$ is metrizable topological vector space and $Y$ is a Fréchet topological vector space (4.2.2).*
- *$Z$ is a complete locally bounded topological vector space and $Y$ is a locally bounded topological vector space (4.2.3).*

This improves several results proven by Cattaneo and Cabello in [Cat80] and [Cab04].

Chapter 5 is devoted to the study of the notion of a cross-section in the context of topological abelian groups and to find situations in which an extension of topological groups $0 \to H \to X \overset{\pi}{\to} G \to 0$ admits a cross-section (i.e. a right inverse for $\pi$) which is continuous or at least continuous at 0. We will finish this chapter using continuous cross-sections to study Problem 2. The main results of this chapter are the following:

- *Let $G, X$ be topological abelian groups and let $\pi : X \to G$ be an open continuous epimorphism. Suppose that $\ker \pi$ is compact and that $G$ is a zero-dimensional $k_\omega$-space; then there exists a continuous cross-section for $\pi$ (5.1.14).*

- $\operatorname{Ext}(G, H) = 0$ *whenever $G$ is the free abelian topological group $A(Y)$ on a zero-dimensional $k_\omega$-space $Y$, and $H$ is a compact abelian group (5.1.15).*

In chapter 6 we discuss Cabello's quasi-homomorphisms of topological abelian groups as well as the stronger notion of *pseudo-homomorphism* which we introduce here. In §6.2 and §6.3 we will explore the connection between quasi-homomorphisms and extensions of topological abelian groups and we will present several cases in which the group Ext can be characterized via the use of quasi-homomorphisms or pseudo-homomorphisms. In §6.4 we will apply the results of the previous two sections to study the quasi-homomorphisms of the form $q : G \to M$, where $G$ is any topological group and $M$ is $\mathbb{R}$ or $\mathbb{T}$. Among others, we prove the following result:

- *Let $M$ be either a Banach space $B$ or the unit circle $\mathbb{T}$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of topological abelian groups such that for every $\alpha < \kappa$, every quasi-homomorphism of $G_\alpha$ to $M$ is approximable. Then every quasi-homomorphism of $\prod_{\alpha<\kappa} G_\alpha$ to $M$ is approximable (6.4.4).*

Finally, in chapter 7 we apply the techniques of the previous chapters to focus on the extensions of topological abelian groups of the form $0 \to M \to X \to G \to 0$ where $M$ is $\mathbb{R}$ or $\mathbb{T}$. In the first section of this chapter we will present several conditions on $G$ that force the splitting of every extension of the previous form (this connects with problems 2 and 3). As a consequence of the results of §7.1 we will deduce that:

- *If $G = \prod_{i\in I} G_i$ is a product of locally precompact abelian groups and $\alpha$ and $\beta$ are arbitrary ordinal numbers,* $\operatorname{Ext}(G, \mathbb{T}^\alpha \times \mathbb{R}^\beta) = 0$ *(7.1.9).*

The last section of chapter 7 is devoted to construct various examples of non-splitting extensions by $\mathbb{R}$ and $\mathbb{T}$. In particular, we will use the weakened topologies on $\mathbb{R}^n$ defined by Stevens ([Ste82]) to construct a group topology $\tau$ such that $\operatorname{Ext}((\mathbb{R}, \tau), \mathbb{R})$ is an infinite dimensional vector space.

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# Chapter 1

# Preliminaries

## §1.1 Abelian groups

**(1.1.1) Terminology.** We will denote by $\omega$ the natural numbers and by $\mathbb{P}$ the prime numbers. As usual, we will denote by $\mathbb{R}$ the group of real numbers, by $\mathbb{Q}$ the group of rational numbers and by $\mathbb{Z}$ the group of integer numbers. The unit circle $\mathbb{T}$ will be considered as the quotient $\mathbb{R}/\mathbb{Z}$.

Since we will deal only with abelian groups we will use additive notation. We will use Greek letters to denote ordinal numbers which will be used as index sets. Given a family of abelian groups $\{G_\alpha : \alpha < \kappa\}$, we will use $\prod_{\alpha<\kappa} G_\alpha$ to denote its *product* and $\bigoplus_{\alpha<\kappa} G_\alpha$ to denote its *direct sum*.

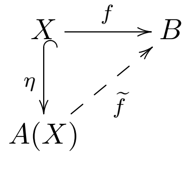
Recall that an abelian group $D$ is *divisible* if for every $m \in \mathbb{Z}$ and $d \in D$ there exists $x \in D$ such that $mx = d$. A group is called torsion (resp. torsion free) if every element has finite (resp. infinite) order. An abelian group $F$ is called *free* if it is a direct sum of infinite cyclic groups.

**(1.1.2) Free abelian group generated by a set.** Given a set $X$, the free abelian group $A(X)$ generated by $X$ is the group of all formal sums of the form

$$n_1 \cdot x_1 + \cdots + n_k \cdot x_k,\ n_i \in \mathbb{Z},\ x_i \in X \ \text{ and } x_i \neq x_j\ \forall i,j \leq k$$

with the natural operation. Notice that $A(X)$ is the direct sum of the infinite cyclic groups generated by the elements of $X$.

Let $B$ be an abelian group and let $f : X \to B$ be a map. Then $\widetilde{f} : A(X) \to B;\ \sum_{i\leq k} n_i x_i \mapsto \sum_{i\leq k} n_i f(x_i)$ is the unique group homomorphism that makes commutative the diagram



where $\eta : X \hookrightarrow A(X)$ is the natural inclusion defined by $\eta(x) = 1 \cdot x$ ([Fuc70, Th. 14.2]).

**(1.1.3) Exact sequences of abelian groups.** A sequence $A_1 \overset{\alpha_1}{\to} A_2 \overset{\alpha_2}{\to} \cdots \overset{\alpha_{n-1}}{\to} A_n$ of abelian groups and homomorphisms is said to be exact if $\alpha_{m-1}(A_{m-1}) = \ker \alpha_m$ for every $2 \leq m \leq n-1$. An exact sequence of the form $E : 0 \to A_1 \to A_2 \to A_3 \to 0$ is called a *short exact sequence* or an *extension of abelian groups*.

*Five-lemma. Suppose that we have the following commutative diagram of abelian groups and homomorphisms in which the horizontal sequences are exact*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

*If $\alpha, \beta, \delta$ and $\varepsilon$ are isomorphisms then $\gamma$ is also an isomorphism (see [ML95, Lemma 3.3 Ch. I]).*

Given two short exact sequences $E : 0 \to H \to X \to G \to 0$ and $E' : 0 \to H \to X' \to G \to 0$, we say that $E$ and $E'$ are *equivalent* if there is a homomorphism $T : X \to X'$ making commutative the following diagram

$$\begin{array}{cccccccccc} E & \quad & 0 & \longrightarrow & H & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0 \\ & & & & \Vert & & \downarrow{\scriptstyle T} & & \Vert & & \\ E' & & 0 & \longrightarrow & H & \longrightarrow & X' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

As a consequence of the Five-lemma if such $T$ exists, it must be an isomorphism. The class of all short exact sequences equivalent to $E$ will be denoted by $[E]$. The extension $E$ is said to *split* if it is equivalent to the trivial sequence $0 \to H \to H \times G \to G \to 0$.

**(1.1.4) Push-out and Pull-back.** [1]The Push-out and Pull-back in the category of abelian groups, can be used to define canonically the *push-out* and *pull-back* sequences associated to a given sequence.

Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ be a short exact sequence of abelian groups and $k : H \to H'$, $t : G' \to G$ two homomorphisms. Then

*(i) The diagram*

$$\begin{array}{cccccccccc} E : & \quad & 0 & \longrightarrow & H & \overset{\imath}{\longrightarrow} & X & \overset{\pi}{\longrightarrow} & G & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle k} & & & & & & \\ & & & & H' & & & & & & \end{array}$$

[1]We will not give the proofs of these facts here in the introduction. Nevertheless, later in §2.2 we will prove a generalized version of these results (for topological abelian groups).

*can be completed to a commutative diagram of the form*

$$\begin{array}{llcccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle s} & & \| & & \\ kE: & 0 & \longrightarrow & H' & \xrightarrow{r} & PO & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

*where $(PO, r, s)$ is the push-out triple of $\imath$ and $k$ in the category of abelian groups, and the bottom sequence $kE$ is a short exact sequence. $kE$ is called the push-out extension of $E$ and $k$ (see [ML95, Lemma 1.4 Ch. III]).*

*(ii) The diagram*

$$\begin{array}{llcccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & & & & & \uparrow{\scriptstyle t} & & \\ & & & & & & & G' & & \end{array}$$

*can be completed to a commutative diagram of the form*

$$\begin{array}{llcccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \| & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \\ Et: & :0 & \longrightarrow & H & \longrightarrow & PB & \xrightarrow[r]{} & G' & \longrightarrow & 0 \end{array}$$

*where $(PB, r, s)$ is pull-back triple of $\pi$ and $t$ in the category of abelian groups, and the bottom sequence $Et$ is a short exact sequence. $Et$ is called the pull-back extension of $E$ and $t$ (see [ML95, Lemma 1.2 Ch. III]).*

*(iii) Using the terminology of (i) and (ii), the extensions $k(Et)$ and $(kE)t$ are equivalent (see [ML95, Lemma 1.6 Ch. III]).*

**(1.1.5) The group** Ext **in the category of abelian groups.** Consider the maps

$$\begin{array}{rccc} \Delta_G: & G & \longrightarrow & G \times G \\ & g & \longmapsto & (g, g) \end{array} \qquad \begin{array}{rccc} \nabla_H: & H \times H & \longrightarrow & H \\ & (h, h') & \longmapsto & h + h' \end{array}$$

Given two short exact sequences $E_1 : 0 \to H \overset{\imath_1}{\to} X_1 \overset{\pi_1}{\to} G \to 0, E_2 : 0 \to H \overset{\imath_2}{\to} X_2 \overset{\pi_2}{\to} G \to 0$ denote by $E_1 \times E_2$ the extension $0 \to H \times H \overset{\imath_1 \times \imath_2}{\to} X_1 \times X_2 \overset{\pi_1 \times \pi_2}{\to} G \times G \to 0$. Using the notation of (1.1.4.i) and (1.1.4.ii), we will define the addition of the equivalence classes $[E_1]$ and $[E_2]$ as

$$[E_1] + [E_2] = \Big[\big(\nabla_H(E_1 \times E_2)\big)\Delta_G\Big] = \Big[\nabla_H\big((E_1 \times E_2)\Delta_G\big)\Big].$$

The second identity follows from (1.1.4.iii). This operation is called the *Baer sum*.

The set $\operatorname{Ext}(G,H)$ of all equivalence classes of short exact sequences of the form $0 \to H \to X \to G \to 0$ endowed with the Baer sum is an abelian group ($[0 \to H \to H \times G \to G \to 0]$ acts as the neutral element and the inverse of a class $[E]$ is $[-\mathrm{Id}_H E]$, see [ML95, Th. 2.1 of Ch. III]).

*(i) If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups and $G$ is an abelian group, there exist exact sequences:*

$$\begin{aligned} 0 &\to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \\ &\to \operatorname{Ext}(C,G) \to \operatorname{Ext}(B,G) \to \operatorname{Ext}(A,G) \end{aligned}$$

$$\begin{aligned} 0 &\to Hom(G,A) \to Hom(G,B) \to Hom(G,C) \\ &\to \operatorname{Ext}(G,A) \to \operatorname{Ext}(G,B) \to \operatorname{Ext}(G,C) \end{aligned}$$

*(see [ML95, Th. 3.2 and 3.4 of Ch. III])*

*(ii) Given an abelian group $G$:*

*(a) $G$ is divisible if and only if $\operatorname{Ext}(X,G) = 0$ for every abelian group $X$ (see [Fuc70, Lemma 52.1 (A)]).*

*(b) $G$ is free if and only if $\operatorname{Ext}(G,X) = 0$ for every abelian group $X$ (see [Fuc70, Lemma 52.1 (B)]).*

*(iii) Let $\{G_\alpha : \alpha < \kappa\}$ be a family of abelian groups and let $X$ be an abelian group. Then for every abelian group $X$:*

$$\operatorname{Ext}\Big(\bigoplus_{\alpha<\kappa} G_\alpha, X\Big) \cong \prod_{\alpha<\kappa} \operatorname{Ext}(G_\alpha, X), \qquad \operatorname{Ext}\Big(X, \prod_{\alpha<\kappa} G_\alpha\Big) \cong \prod_{\alpha<\kappa} \operatorname{Ext}(X, G_\alpha)$$

*(see [ML95, Exercise 3 of Sec.8 Ch. III])*

An abelian group $G$ that satisfies that $\operatorname{Ext}(X,G) = 0$ for every torsion free abelian group $X$ is called a *cotorsion group* ([Fuc70, §54]).

## §1.2 General topology

**(1.2.1) Terminology.** We will follow the notation and terminology of [Eng89]. We will denote a topology defined on a set $X$ by $\tau$. Given a subset $A$ of a topological space $(X,\tau)$, we will denote by $\tau_{|A}$ the topology induced by $\tau$ on $A$ i.e. the one whose open sets are of the form $O \cap A$ with $O \in \tau$.

Given two topologies $\tau$ and $\tau'$ on a set $X$ if $\tau \subset \tau'$ we will write $\tau \leq \tau'$ and we will say that:

- $\tau'$ is *finer (stronger)* than $\tau$.
- $\tau$ is *coarser (weaker)* than $\tau'$.

A family of non-empty sets $B \subset \tau$ is called a *base for a topological space* $(X, \tau)$ if every non-empty open subset of $X$ can be represented as the union of a subfamily of $B$.

A neighborhood of a point $x$ in a topological space $(X, \tau)$ is a set $U \subset X$ such that $x \in O \subset U$ for some $O \in \tau$. We will use $\mathcal{N}_x(X)$ to denote the set of all neighborhoods of $x$ in $(X, \tau)$. A family $B(x)$ of neighborhoods of $x$ is called a *base for a topological space* $(X, \tau)$ *at the point* $x$ (or a *system of neighborhoods of* $x$) if for any neighborhood $V$ of $x$ there exists $U \in B(x)$ such that $x \in U \subset V$.

The adherence of a set $A$ will be denoted by $\overline{A}$ and the interior by $\mathring{A}$.

Let $X$ be a Hausdorff topological space. Here we recall some topological properties that we will use in the text:

- We will say that $X$ is *zero-dimensional* if each point $x \in X$ admits a basis $B(x)$ consisting of closed and open sets (*clopen sets*).
- $X$ is called *paracompact* if every open cover of $X$ has a locally finite refinement[2]. Metric spaces and compact spaces are paracompact (see [Eng89, 5.1.1 and 5.1.3]).
- $X$ is *totally disconnected* if it has no non-trivial connected subsets. Discrete spaces and zero-dimensional $T_2$-spaces are totally disconnected (see [Eng89, p. 369].
- A Tychonov space $X$ is *Čech-complete space* if it is a $G_\delta$-set in its Čech-Stone compactification $\beta X$ ([Eng89, §3.9]). Locally compact spaces are Čech- complete, complete metrizable spaces are Čech-complete ([Eng89, 4.3.26])

**(1.2.2) Initial and final topologies.** Let $X$ be a set and let $\mathcal{H} = \{f_\alpha : X \to Y_\alpha : \alpha < \kappa\}$ be a family of maps where $Y_\alpha$ is a topological space for every $\alpha < \kappa$. The *initial (or weak)* topology $\tau_{\mathcal{I}}$ on $X$ is the coarsest topology on $X$ that makes continuous all the maps in $\mathcal{H}$. The initial topology $\tau_{\mathcal{I}}$ has the following properties:

- The family of all finite intersections of sets of the form $f_\alpha^{-1}(O_\alpha)$ where $O_\alpha$ is open in $Y_\alpha$, constitutes a base for $\tau_{\mathcal{I}}$ ([Wil70, 8.9]).
- Given a topological space $Y$, a map $f : Y \to (X, \tau_{\mathcal{I}})$ is continuous if and only if $f_\alpha \circ f$ is continuous for every $\alpha < \kappa$ ([Wil70, 8.10]).

[2]Recall that a family $\{A_\alpha : \alpha < \kappa\}$ is locally finite if for every $x \in X$ and $U \in \mathcal{N}_x(X)$, the set $\{\alpha < \kappa : A_\alpha \cap U \neq \emptyset\}$ is finite

Let $\mathcal{A} = \{a_\alpha : Y_\alpha \to X;\ \alpha < \kappa\}$ be a family of maps where $Y_\alpha$ is a topological space for every $\alpha < \kappa$ . The *final (or strong)* topology $\tau_{\mathcal{F}}$ on $X$ is the finest topology on $X$ that makes all the maps of $\mathcal{A}$ continuous. The final topology $\tau_{\mathcal{F}}$ has the following properties:

- A subset $O \subset X$ is open in $\tau_{\mathcal{F}}$ if and only if $a_\alpha^{-1}(O)$ is open in $Y_\alpha$ for all $\alpha < \kappa$ ([Wil70, 9H]).
- Given a topological space $Y$, a map $a : (X, \tau_{\mathcal{F}}) \to Y$ is continuous if and only if $a \circ a_\alpha$ is continuous for every $\alpha < \kappa$ ([Wil70, 9H]).

**(1.2.3) Open maps.** Recall that a map $f : X \to Y$ between topological spaces is open if $f(O)$ is open in $Y$ for every $O$ open in $X$. $f$ is said to be *relatively open* if its corestriction $X \to f(X);\ x \mapsto f(x)$ is open.

**(1.2.4) Perfect mappings.** Let $X$ and $Y$ be topological spaces. Suppose that $X$ is Hausdorff. A continuous map $f : X \to Y$ is *perfect* if it is closed and all fibers $f^{-1}(y)$ are compact subsets of $X$. Some well-known facts concerning perfect mappings are the following:

*(i) The inverse image of a compact set by a perfect mapping is compact* ([Eng89, Th. 3.7.1])*. A perfect map is one-to-one if and only if it is an embedding.*

*(ii) Given topological spaces $X, Y, Y'$, if $f : X \to Y$ is a perfect map and $g : X \to Y'$ is any continuous map then their diagonal product $f \triangle g : X \to Y \times Y';\ x \mapsto (f(x), g(x))$ is perfect* ([Eng89, Th. 3.7.10])*.*

**(1.2.5) Inverse limits of topological spaces.** Suppose that $X_\alpha$ is a topological space for every $\alpha < \kappa$ and that $\{\pi_{\alpha,\beta} : X_\alpha \to X_\beta : \beta < \alpha < \kappa\}$ is a family of continuous mappings satisfying that $\pi_{\alpha,\alpha} = \mathrm{Id}_{X_\alpha}$ and $\pi_{\alpha,\gamma} \circ \pi_{\gamma,\beta} = \pi_{\alpha,\beta}$ for every $\beta < \gamma < \alpha$. Then the family $\mathcal{P} = \{X_\alpha, \pi_{\alpha,\beta} : \beta < \alpha < \kappa\}$ is called an *inverse system*[3]. The space

$$\lim_{\leftarrow} \mathcal{P} = \{(x_\alpha)_{\alpha<\kappa} : x_\beta = \pi_{\alpha,\beta}(x_\alpha)\ \forall \beta < \alpha < \kappa\} \subset \prod_{\alpha<\kappa} X_\alpha$$

is called the *inverse limit of the system* $\mathcal{P}$ and it is a closed subspace of $\prod_{\alpha<\kappa} X_\alpha$.

It is known that if for every $\alpha < \kappa$, $B_\alpha$ is a base for $X_\alpha$ and $\pi_\alpha : \prod_{\gamma<\kappa} X_\gamma \to X_\alpha$ is the canonical projection then $\{\lim_{\leftarrow} \mathcal{P} \cap \pi_\alpha^{-1}(U_\alpha) : U_\alpha \in B_\alpha\}$ is a base for $\lim_{\leftarrow} \mathcal{P}$ *(see [Eng89, Prop 2.5.5]).*

[3] The usual definition of an inverse system does not require the index set to be well ordered. Nevertheless, we will use this less general definition because it suits our purposes.

Suppose that $\{f_\alpha : Y \to X_\alpha : \alpha < \kappa\}$ is a family of continuous maps such that $\pi_{\alpha,\beta} \circ f_\alpha = f_\beta \ \forall \beta < \alpha < \kappa$. Then the map $f : Y \to \lim_{\leftarrow} \mathcal{P};\ y \mapsto (f_\alpha(y))_{\alpha<\kappa}$ is continuous (see [Dug66, Th. 2.5 Appendix 2]).

**(1.2.6) $k_\omega$-spaces.** A Hausdorff topological space $X$ is called a *$k_\omega$-space*, if it has an increasing sequence of compact subspaces $\{K_n : n < \omega\}$ such that $X = \bigcup_{n<\omega} K_n$ and a subset $A$ is closed in $X$ if and only if $A \cap K_n$ is closed for every $n < \omega$ (see [FS77]). Given another topological space $Y$, a map $f : X \to Y$ is continuous if and only if $f_{|K_n} : K_n \to Y$ is continuous for every $n < \omega$.

**(1.2.7) Character and pseudocharacter.** Let $X$ be a Hausdorff topological space. The *character* of a point $x \in X$ is the smallest cardinal number of the form[4] $|B(x)|$, where $B(x)$ is a base for $X$ at the point $x$. The *pseudocharacter of* $x$ is the smallest cardinal number of the form $|\mathcal{U}|$ where $\mathcal{U}$ is a family of open subsets of $X$ such that $\bigcap_{U\in\mathcal{U}} U = \{x\}$.

If a compact space has countable pseudocharacter, then it is first countable. This is because for compact spaces character and pseudocharacter coincide ([Eng89, 3.1.F]).

## §1.3 Topological abelian groups

**(1.3.1) Terminology.** All the topological groups will be Hausdorff.

Let $G$ be an abelian group and $\tau$ a topology on $G$. We will say that $(G, \tau)$ is a *topological abelian group* if the maps

$$\begin{array}{ccc} (G,\tau)\times(G,\tau) & \longrightarrow & (G,\tau) \\ (x,y) & \longmapsto & x+y \end{array} \qquad \begin{array}{ccc} (G,\tau) & \longrightarrow & (G,\tau) \\ x & \longmapsto & -x \end{array}$$

are continuous. In this situation $\tau$ will be called a *group topology*.

Let $\mathcal{N}$ be a family of subsets of an abelian group $G$ such that $0 \in U \ \forall U \in \mathcal{N}$. The family $\mathcal{N}$ is a system of open neighborhoods of 0 for a group topology on $G$ if and only if it satisfies the following properties (see [HR62, Th. 4.5]):

(a) For every $U, V \in \mathcal{N}$ there exists another $W \in \mathcal{N}$ with $W \subset U \cap V$.

(b) For every $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ with $-V \subset U$.

(c) For every $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ with $V + V \subset U$.

(d) For every $U \in \mathcal{N}$ and $g \in U$, there exists $V \in \mathcal{N}$ with $g + V \subset U$.

---

[4] Given a set $A$, we will use $|A|$ to denote its cardinal.

**(1.3.2) Group topologies induced by group-norms.** A *group-norm* on an abelian group $G$ is a function $\nu : G \to \mathbb{R}$, satisfying the following conditions for all $x, y \in G$:

(a) $\nu(x) \geq 0$,

(b) $\nu(x) = 0$ if and only if $x = 0$,

(c) $\nu(x + y) \leq \nu(x) + \nu(y)$,

(d) $\nu(x) = \nu(-x)$.

The family $\mathcal{B} = \{x + U_\varepsilon : \varepsilon \in \mathbb{R}, \varepsilon > 0, x \in G\}$, where $U_\varepsilon = \{x \in G : \nu(x) < \varepsilon\}$, is a basis for a metrizable group topology on $G$ that we will denote by $\tau_\nu$. The topology $\tau_\nu$ is called the group topology induced by the group-norm $\nu$.

Notice that the topology on a topological abelian group $G$ is induced by a group-norm if and only if it is induced by an invariant metric (i.e. a metric $d : G \times G \to \mathbb{R}$ such that $d(x+a, y+a) = d(x, y)\ \forall a, x, y \in \mathbb{R}$). Furthermore, a topological group has a countable basis of open neighborhoods at 0 if and only if it has a topology induced by an invariant metric (and therefore a group-norm) ([HR62, 8.3]).

**(1.3.3) Initial and final group topologies.** Let $G$ be an abelian group and let $\mathcal{H} = \{f_\alpha : G \to H_\alpha : \alpha < \kappa\}$ be a family of homomorphisms where $H_\alpha$ is a topological abelian group for every $\alpha < \kappa$. The *initial (or weak)* group topology $\tau_{\mathcal{I}}$ on $G$ induced by $\mathcal{H}$ is the coarser group topology on $G$ that makes continuous all the homomorphisms of $\mathcal{H}$. The initial group topology $\tau_{\mathcal{I}}$ has the following properties:

- The family of all finite intersections of sets of the form $f_\alpha^{-1}(O_\alpha)$ where $O_\alpha$ is open in $H_\alpha$, conforms a base for $\tau_{\mathcal{I}}$.
- Given a topological abelian group $H$, a homomorphism $f : H \to (G, \tau_{\mathcal{I}})$ is continuous if and only if $f_\alpha \circ f$ is continuous for every $\alpha < \kappa$.

Let $\mathcal{A} = \{a_\alpha : H_\alpha \to G;\ \alpha < \kappa\}$ be a family of homomorphisms where $H_\alpha$ is a topological abelian group for every $\alpha < \kappa$ . The *final (or strong)* group topology $\tau_{\mathcal{F}}$ on $G$ is the finest group topology on $G$ that makes continuous all the homomorphisms of $\mathcal{A}$. Given a topological abelian group $H$, a homomorphism $a : (G, \tau_{\mathcal{F}}) \to H$ is continuous if and only if $a \circ a_\alpha$ is continuous for every $\alpha < \kappa$.

It is easily seen that the final group topology induced by $\mathcal{A}$ on $G$ does not coincide in general with the final topology induced by $\mathcal{A}$ (defined as in (1.2.2)).

**(1.3.4) Group topologies on the direct sum.** Let $\{G_\alpha : \alpha < \kappa\}$ be family of topological abelian groups. The *box topology* on the direct sum

$\bigoplus_{\alpha<\kappa} G_\alpha$ is the group topology induced by the system of neighborhoods

$$\Big\{\big(\prod_{\alpha<\kappa} U_\alpha\big) \cap \bigoplus_{\alpha<\kappa} G_\alpha : U_\alpha \in \mathcal{N}_0(G_\alpha)\ \forall \alpha < \kappa\Big\}.$$

The *coproduct topology* on $\bigoplus_{\alpha<\kappa} G_\alpha$ is the final group topology with respect to the natural inclusions $G_\gamma \hookrightarrow \bigoplus_{\alpha<\kappa} G_\alpha$.

*(i) The coproduct topology on $\bigoplus_{\alpha<\kappa} G_\alpha$ is the finest group topology that induces the original topologies on all groups $G_\alpha$* ([CD03, Cor. 10]).

*(ii) If $\kappa = \omega$ the box topology and the coproduct topology on $\bigoplus_{\alpha<\kappa} G_\alpha$ coincide* ([CD03, Prop. 11]).

**(1.3.5) Locally compact abelian groups.** Locally compact abelian groups are those that contain a compact neighborhood of the neutral element. The class of locally compact abelian groups will be denoted by $\mathcal{L}$, and our basic reference for this class will be [HR62]. It is known that *any $G \in \mathcal{L}$ is topologically isomorphic to $\mathbb{R}^n \times G_0$ where $n < \omega$ and $G_0 \in \mathcal{L}$ has a compact open subgroup* (This is the structure theorem for locally compact abelian group, see [HR62, 24.30]).

**(1.3.6) Locally precompact abelian groups.** A topological abelian group is called *locally precompact* if it can be embedded as dense subgroup of a locally compact abelian group. $\mathbb{Q}$ is an example of a locally precompact abelian group that is not locally compact.

**(1.3.7) $p$-adic numbers.** Given $p \in \mathbb{P}$, *the group of $p$-adic numbers* $\mathbb{Q}_p$ is defined as the group of formal series of the form

$$\sum_{n\in\mathbb{Z}} x_n p^n \quad (x_n \in \{0,\dots,p-1\}\ \forall n \in \mathbb{Z} \text{ and } \exists k \in \mathbb{Z} : x_n = 0\ \forall n < k)$$

endowed with the natural addition.

The family of subgroups

$$H_k = \Big\{\sum_{n\in\mathbb{Z}} x_n p^n \in \mathbb{Q}_p : x_n = 0\ \forall n < k\Big\} \leq \mathbb{Q}_p, \quad k \in \mathbb{Z}$$

conforms a system of neighborhoods of $0 = \sum 0\cdot p^n$ for a locally compact, totally disconnected group topology on $\mathbb{Q}_p$ in which the $H_k$'s are compact. Furthermore, the $H_k$'s are the only proper closed subgroups of $\mathbb{Q}_p$ (see [HR62, 10.16]) which implies that all the non-trivial proper closed subgroups of $\mathbb{Q}_p$ are open. The group of *$p$-adic integers* is defined as $\mathbb{Z}_p = H_0$.

**(1.3.8) Pontryagin duality.** Let $G$ be a topological abelian group and let $\mathrm{CHom}(G,\mathbb{T})$ be the group of all continuous homomorphisms from $G$

to $\mathbb{T}$, with the pointwise operation. The elements of $\mathrm{CHom}(G,\mathbb{T})$ are called *continuous characters* and the dual group $G^\wedge$ is defined as the abelian group $\mathrm{CHom}(G,\mathbb{T})$ endowed with the compact-open topology. Recall that the family of subsets

$$P(F,r) = \{\chi \in \mathrm{CHom}(G,\mathbb{T}) : \chi(F) \subset (-r,r) + \mathbb{Z}\} \subset \mathrm{CHom}(G,\mathbb{T})$$

where $F \subset G$ is compact and $r \in \mathbb{R}$, constitutes a basis of open neighborhoods of 0 for the compact open topology on $\mathrm{CHom}(G,\mathbb{T})$. The bidual group $G^{\wedge\wedge}$ is defined as $(G^\wedge)^\wedge$.

A subgroup $H$ of a topological abelian group $G$ is said to be *dually closed* if for every element $x$ of $G\backslash H$ there is $\chi \in G^\wedge$ such that $\chi(H) = \{0 + \mathbb{Z}\}$ and $\chi(x) \neq 0 + \mathbb{Z}$. The subgroup $H$ is said to be *dually embedded* if every continuous character defined on $H$ can be extended to a continuous character on $G$.

We have the following facts:

*(a) If G is in $\mathcal{L}$, the map*

$$\begin{array}{ccc} G & \longrightarrow & G^{\wedge\wedge} \\ g & \longmapsto & \big(ev_g : G^\wedge \to \mathbb{T};\ \chi \mapsto \chi(g)\big) \end{array}$$

*is a topological isomorphism (this is the celebrated Pontryagin duality theorem, see [HR62, 24.8])*

*(b) G is compact (resp. discrete) if and only if $G^\wedge$ is discrete (resp. compact) (see [HR62, 23.17]).*

*(c) $\mathbb{Z}^\wedge \cong \mathbb{T}$, $\mathbb{T}^\wedge \cong \mathbb{Z}$ and $\mathbb{R}^\wedge \cong \mathbb{R}$ (see [HR62, 23.27(a) and 23.27(e)]).*

**(1.3.9) MAP groups.** A topological abelian group $G$ is called *maximally almost periodic* (shortly MAP) if for every $x \in G\backslash\{0\}$ there exist $\chi \in G^\wedge$ with $\chi(x) \neq 0+\mathbb{Z}$. MAP groups are also called *groups with sufficiently many characters.*

*(i) Locally compact abelian groups are MAP ([HR62, 22.17]).*

*(ii) Compact subgroups of Hausdorff MAP abelian groups are dually embedded and dually closed ([BMP96, Prop. 1.4]).*

**(1.3.10) Locally quasi-convex topological abelian groups.** A subset $A$ of a topological abelian group $G$ is called *quasi-convex* if for every $x \in G \setminus A$ there is a $\chi \in G^\wedge$ such that $\chi(a) \in [-1/4, 1/4] + \mathbb{Z}$ $\forall a \in A$ and $\chi(x) \notin [-1/4, 1/4] + \mathbb{Z}$. A topological abelian group is called *locally quasi-convex* if it has a neighborhood basis of 0 consisting of quasi-convex sets.

**(1.3.11) Raĭkov Completeness.** A topological abelian group $G$ is *Raĭkov-complete* (shortly *complete*) if every Cauchy filter in $G$ converges (see [AT08, Section 3.6]).

For every topological abelian group $G$ there exists a Raĭkov-complete topological abelian group $\varrho G$ such that $G$ is a dense subgroup of $\varrho G$. $\varrho G$ is called the *Raĭkov completion* of $G$ and is unique in the sense that for every Raĭkov-complete topological abelian group $G^*$ containing $G$ as a dense subgroup, there exists a topological isomorphism $\phi : G^* \to \varrho G$ such that $\phi(g) = g\ \forall g \in G$ (see [AT08, Th. 3.6.14]).

Every continuous homomorphism of topological abelian groups $f : G \to H$ can be extended to a unique continuous homomorphism $\varrho f : \varrho G \to \varrho H$ ([AT08, Cor. 3.6.17]).

A topological group $G$ is locally precompact if and only if its Raĭkov completion is locally compact.

**(1.3.12) Almost-metrizability.** Recall that a topological abelian group $G$ is *almost-metrizable* if it contains a compact subgroup $K$ such that $G/K$ is metrizable.

*(i) A topological group is Čech-complete if and only if it is almost-metrizable and Raĭkov-complete (see [AT08, Theo. 4.3.15]).*

*(ii) Metrizable topological abelian groups and locally compact groups are almost-metrizable.*

**(1.3.13) Three space properties in topological abelian groups.** A property $(\mathcal{P})$ is called a *three space property in the category of topological abelian groups* when given any topological abelian group $G$ and any closed subgroup $H \leq G$, if $H$ and $G/H$ have the property $(\mathcal{P})$ then $G$ has the property $(\mathcal{P})$.

The following are examples of three space properties in this category:

- Metrizability ([HR62, 5.38(e)]).
- Raĭkov completeness ([War89, Exercise 5.1]).
- Compactness and local compactness ([HR62, Th. 5.25]).

However, $\sigma$-compactness, sequential completeness and realcompactness are not three space properties (see [BT06] for more examples).

**(1.3.14) Free abelian topological groups.** Let $X$ a completely regular Hausdorff topological space. The free abelian topological group over $X$ is the free abelian group $A(X)$ endowed with the unique Hausdorff group topology that satisfies the following properties:

- The mapping $\eta : X \hookrightarrow A(X);\ x \mapsto 1 \cdot x$, becomes a topological embedding.
- For every continuous mapping $f : X \to G$, where $G$ is an abelian topological group, the group homomorphism $\widetilde{f} : A(X) \to B;\ \sum_{i \leq k} n_i x_i \mapsto \sum_{i \leq k} n_i f(x_i)$ is continuous.

(see [Aus99, Th. 12.1] and [AT08, §7]).

## §1.4 Topological vector spaces

**(1.4.1) Terminology.** Our basic reference for topological vector spaces will be [Sch86]. We will only consider vector spaces over the field $\mathbb{R}$.

Let $X$ be a vector space and $\tau$ a topology on $X$. We will say that $(X, \tau)$ is a *topological vector space* if the maps

$$\begin{array}{ccc} (X,\tau)\times(X,\tau) & \longrightarrow & (X,\tau) \\ (x,y) & \longmapsto & x+y \end{array} \qquad \begin{array}{ccc} \mathbb{R}\times(X,\tau) & \longrightarrow & (X,\tau) \\ (r,x) & \longmapsto & r\cdot x \end{array}$$

are continuous. In this situation $\tau$ will be called a *vector space topology*.

Normed spaces are topological vector spaces. Given a normed space $X$, we will use $B(x, \delta)$ to denote the open ball centered at $x \in X$ with radius $\delta > 0$.

Recall that a subset $A$ of a topological vector space $X$ is *bounded* if $\forall U \in \mathcal{N}_0(X)$, there exists $\lambda \in \mathbb{R}$ such that $A \subset \lambda U$ ([Sch86, §5 Ch. I]). A subset $B \subset X$ is called *convex* if for every $x, y \in B$, $\{\lambda x + (1-\lambda)y : 0 < \lambda < 1\} \subset B$ ([Sch86, §1 Ch. II]). Locally bounded (resp. locally convex) topological vector spaces are those that admit a system of bounded (resp. convex) neighborhoods of the neutral element.

The Raĭkov completion of a topological vector space $X$ is a topological vector space that contains $X$ as a dense subspace (see [Sch86, 1.5 Ch. 1]).

A topological vector space is called *Fréchet* if it is metrizable, (Raĭkov) complete and locally convex. Complete normed spaces are called *Banach* spaces.

**(1.4.2) Three space properties in topological vector spaces.** A property $(\mathcal{P})$ is called a *three space property in the category of topological vector spaces* when given any topological vector space $X$ and any closed subspace $Y \leq X$, if $Y$ and $X/Y$ have the property $(\mathcal{P})$ then $X$ has the property $(\mathcal{P})$. If a property $(\mathcal{P})$ is a three space property in the category of topological abelian groups then it is also a three space property in the category of topological vector spaces.

Local boundedness is a three space property in the category of topological vector spaces ([RD81a, Th. 3.2]). Nevertheless, local convexity is not a three space property in this category (see [KPR84, Chapter 5].)

**(1.4.3) $\ell^p$.** Given $x = (x_n)_{n\in\omega} \in \mathbb{R}^\omega$, and $p \in (0, \infty)$ we define $\|x\|_p = (\sum_{n\in\omega}^{\infty} |x_n|^p)^{1/p}$. The space $\ell^p$ is defined as the subspace $\{x \in \mathbb{R}^\omega : \|x\|_p < \infty\} \leq \mathbb{R}^\omega$ endowed with the topological vector space structure induced by $\|\cdot\|_p$. For $p \geq 1$, $\ell^p$ is a Banach space.

# Chapter 2

# Extensions of topological abelian groups

Throughout this chapter we will develop the basis of the theory of extensions of topological abelian groups. The notions of *push-out* and *pull-back* extension (§2.2) will give us a crucial device to obtain new extensions from others and will be the key for our future constructions. Notice that the objects and results presented in this chapter generalize the theory of extensions of abelian groups ((1.1.3), (1.1.4))

## §2.1 Equivalence and splitting

**(2.1.1) Definitions.** A short exact sequence of topological abelian groups $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ is called *an extension of topological abelian groups* if the maps $\imath$ and $\pi$ are relatively open (see (1.2.3)), continuous homomorphisms. If there is no place to confusion, we will often abbreviate this by saying that $E$ is an *extension*.

Notice that the exactness of $E$ implies that $\imath(H) = \ker \pi$ is closed. Furthermore, according to the first isomorphism theorem ([HR62, (5.27)]), $X/\imath(H)$ is topologically isomorphic to $G$.

Two extensions of topological abelian groups $E : 0 \to H \to X \to G \to 0$ and $E' : 0 \to H \to X' \to G \to 0$ are said to be *equivalent* if there exists a continuous homomorphism $T : X \to X'$ making the squares in (E1) commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & X' & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{E1}$$

Under these conditions we will say that $T$ *witnesses the equivalence* of $E$

and $E'$ and we will write $E \equiv E'$. We will denote by $[E]$ the class of all extensions of topological abelian groups equivalent to $E$.

**(2.1.2) Lemma.** Let $H$ be a subgroup of a group $G$ and let $\tau$ and $\tau'$ be (not necessary Hausdorff) group topologies on $G$ such that both coincide in $H$ (i.e. $\tau'_{|H} = \tau_{|H}$), $\tau' \leq \tau$ and $(G, \tau)/H = (G, \tau')/H$. Then $\tau' = \tau$.

*Proof.* See [Roe72]. ∎

We will show now that the continuous homomorphism $T$ of the above definition is in fact a topological isomorphism.

**(2.1.3) Proposition.** *Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ and $E' : 0 \to H \overset{\imath'}{\to} X' \overset{\pi'}{\to} G \to 0$ be extensions of topological abelian groups. Suppose that a continuous homomorphism $T : X \to X'$ makes the squares in the following diagram commutative*

$$\begin{array}{lccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \Vert & & \downarrow T & & \Vert & & \\ E': & 0 & \longrightarrow & H & \xrightarrow[\imath']{} & X' & \xrightarrow[\pi']{} & G & \longrightarrow & 0 \end{array}$$

*Then $T$ is a topological isomorphism.*

*Proof.* The five-lemma (1.1.3) implies that $T$ is an isomorphism of abelian groups. Let $\mathcal{T}$ be the group topology on $X'$ induced by the isomorphism $T$ and $\tau$ the original topology on $X'$. Notice that $\mathcal{T}$ is finer that $\tau$. Denote by $\imath_{\mathcal{T}} : H \to (X', \mathcal{T})$ and $\pi_{\mathcal{T}} : (X', \mathcal{T}) \to G$ the maps defined by $\imath_{\mathcal{T}}(h) = \imath'(h)$ and $\pi_{\mathcal{T}}(x) = \pi'(x)$ respectively. Since $\imath_{\mathcal{T}} : H \to (X', \mathcal{T})$ and $\imath' : H \to (X', \tau)$ are embeddings $\tau_{|\imath'(H)} = \mathcal{T}_{|\imath'(H)}$. $\pi_{\mathcal{T}} : (X', \mathcal{T}) \to G$ and $\pi' : (X', \tau) \to G$ are open and continuous homomorphisms. Consequently, $(X', \mathcal{T})/\imath'(H) = (X', \tau)/\imath'(H)$ and according to (i), $\mathcal{T} = \tau$. This proves that $T$ is a topological isomorphism. ∎

**(2.1.4) Remark.** Notice that a property $(\mathcal{P})$ is a three space property in the category of topological abelian groups (1.3.13) if and only if given any extension of topological abelian groups of the form $0 \to H \to X \to G \to 0$ in which $H$ and $G$ have $(\mathcal{P})$, then $X$ also has the property $(\mathcal{P})$.

**(2.1.5) Splitting of extensions of topological abelian groups.** Let $G$ and $H$ be topological abelian groups and let $\imath_H : H \to H \times G$ and $\pi_G : H \times G \to G$, be the canonical maps. Considering on $G \times H$ the product topology, the short exact sequence $E_0 : 0 \to H \overset{\imath_H}{\to} H \times G \overset{\pi_G}{\to} G \to 0$ is trivially an extension of topological abelian groups which is called the *trivial extension* of $G$ by $H$.

We say that the extension of topological abelian groups $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ *splits* if it is equivalent to the trivial extension $E_0$.

Given a topological abelian group $A$ and a subgroup $B \leq A$, $B$ is said to *split from*[1] $A$ if there exists another subgroup $C \leq A$ such that the natural homomorphism $\phi : B \times C \to A$; $(b, c) \mapsto b + c$ is a topological isomorphism.

*(i) $E$ splits if and only if $\imath(H)$ splits from $X$.*

*Proof.* Suppose that $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ splits and consider $T : H \times G \to X$ the topological isomorphism witnessing the equivalence between $E$ and the trivial extension $E_0 : 0 \to H \overset{\imath_H}{\to} H \times G \overset{\pi_G}{\to} G \to 0$. The map $\phi : \imath(H) \times T(\{0\} \times G) \to X$; $(\imath(h), T(0, g)) \mapsto \imath(h) + T(0, g) = T(h, g)$ is a topological isomorphism.

Conversely, suppose that there exists a subgroup $Y \leq X$ such that the map $\varphi : \imath(H) \times Y \to X$; $(\imath(h), y) \mapsto \imath(h) + y$ is a topological isomorphism. Write $\varphi^{-1} = (f_1, f_2) : X \to \imath(H) \times Y$ and define $t : X \to H \times G$ as $t(x) = (\imath^{-1}(f_1(x)), \pi(f_2(x)))$. One can easily prove that $t \circ \imath = \imath_H$. Furthermore, since $\pi \circ f_2(x) = \pi(x)$, we obtain that $\pi_G \circ t = \pi$. Consequently $t$ witnesses the equivalence of $E$ and $E_0$. □

$E$ is said to *split algebraically* (or to be *algebraically splitting*) if, regarded as a short exact sequence of abelian groups, it splits (in the sense of (1.1.3)). Now we introduce a criterion that will be very useful in the following sections:

*(ii) An extension of topological abelian groups $E : 0 \to H \to X \to G \to 0$ is algebraically splitting if and only if it is equivalent to an extension of topological abelian groups of the form $E_\tau : 0 \to H \overset{\imath_\tau}{\to} (H \times G, \tau) \overset{\pi_\tau}{\to} G \to 0$ where $\tau$ is a group topology (not necessarily the product topology) on $H \times G$ and $\imath_\tau, \pi_\tau$ are the canonical mappings.*

*Proof.* Let $T : X \to H \times G$ be an isomorphism of abelian groups witnessing the algebraic equivalence of $E$ and the trivial extension. Define $\tau$ as the group topology on $H \times G$ induced by $T$ i.e. the group topology that has $\{T(U) : U \in \mathcal{N}_0(X)\}$ as a system of neighborhoods of 0. The converse implication is trivial. □

**(2.1.6) Example.** Let $D$ be any dense subgroup of $\mathbb{R}$ of the form $\mathbb{Z} + a\mathbb{Z}$ for some $a \notin \mathbb{Q}$ (to check that these groups are dense in $\mathbb{R}$ see [Mor77, Cor.1, Prop. 23 §2]). Take $\pi : D \to D/\mathbb{Z}$; $d \to d + \mathbb{Z}$, the natural projection. The sequence $E : 0 \to \mathbb{Z} \hookrightarrow D \overset{\pi}{\to} D/\mathbb{Z} \to 0$ is an extension of topological abelian groups. Notice that by definition of $D$, we see that $E$ splits algebraically.

Let us see that the extension $E$ does not split. Suppose that $E$ splits. By definition of splitting extension of topological abelian groups, $D$ is topologically isomorphic to $\mathbb{Z} \times L$ for some $L \leq \mathbb{R}$. This implies that taking

[1] Some authors express this by saying that $B$ *splits topologically*, from $A$ or that $B$ is a *topological direct summand* of $A$.

Raĭkov completions,

$$\mathbb{R} \cong \varrho D \cong \varrho(\mathbb{Z} \times L) \cong \mathbb{Z} \times \varrho L,$$

which gives us a contradiction because $\mathbb{R}$ is not algebraically isomorphic (as an abelian group) to the product $\mathbb{Z} \times \varrho L$.

**(2.1.7) Proposition.** *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological groups. The following are equivalent:*

*(i) $E$ splits.*

*(ii) There exists a continuous homomorphism $S : G \to X$ with $\pi \circ S = \mathrm{id}_G$.*

*(iii) There exists a continuous homomorphism $P : X \to H$ with $P \circ \imath = \mathrm{id}_H$.*

*Proof.* *(i)* $\Rightarrow$ *(ii)*. If $E$ splits, there exists a continuous homomorphism $T : H \times G \longrightarrow X$ making the squares in (E2) commutative

$$\begin{array}{ccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \| & & \uparrow T & & \| & & \\ E_0: & 0 & \longrightarrow & H & \xrightarrow[\imath_H]{} & H \times G & \xrightarrow[\pi_G]{} & G & \longrightarrow & 0 \end{array} \tag{E2}$$

where $i_H$ and $\pi_G$ are the canonical maps. Define $S : G \to X$ as $S(g) = T(0, g)$. Using the commutativity of the right square (E2), $\pi(S(g)) = \pi(T(0, g)) = \pi_G(0, g) = \mathrm{Id}_G(g)$.

*(ii)* $\Rightarrow$ *(iii)*. Since for every $x \in X$, $x - S \circ \pi(x) \in \ker \pi$, the map $P : X \to H$; $x \mapsto \imath^{-1}(x - S \circ \pi(x))$ is a well-defined continuous homomorphism that satisfies the required property.

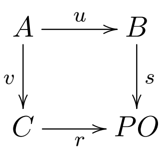
*(iii)* $\Rightarrow$ *(i)*. Define $t : X \to H \times G$ as $t(x) = (P(x), \pi(x))$. Since $t \circ \imath(h) = (h, 0) = \imath_H(h)$ and $\pi_G \circ t(x) = \pi(x)$, it follows that $t$ witnesses the equivalence between $E$ and the trivial extension $E_0$. ∎

## §2.2 Push-out and pull-back extensions

**(2.2.1) Push-out square.** Given topological abelian groups $A$, $B$ and $C$ and continuous homomorphisms $u : A \to B$ and $v : A \to C$, we will call Push-out (briefly $PO$) the quotient of the product $B \times C$ by the subgroup $\Delta = \{(-u(a), v(a)) : a \in A\}$ ([2]). If we consider the maps $r : C \to PO$; $c \mapsto$

[2] $PO$ is endowed with the quotient topology of the corresponding product topology on $B \times C$

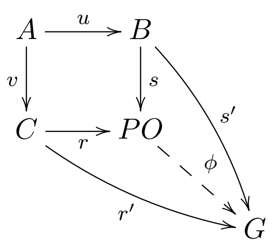
$(0, c) + \Delta$ and $s : B \to PO$; $b \mapsto (b, 0) + \Delta$ we obtain the following commutative diagram:




The triple $(PO, s, r)$ is called the *push-out* of $u$ and $v$.

The push-out satisfies the following universal property:

*(i) For every topological abelian group $G$ and continuous homomorphisms $r' : C \longrightarrow G$, $s' : B \longrightarrow G$ with $s' \circ u = r' \circ v$, there is a unique continuous homomorphism $\phi$ from $PO$ to $G$ making the diagram (E3) to commute.*




(E3)

*Proof.* Define $\phi : PO = (B \times C)/\Delta \to G$ as $\phi((b, c) + \Delta) = s'(b) + r'(c)$. The identities (E7) and (E5) imply that $\phi$ is a well-defined homomorphism.

$$\begin{aligned}
\phi\big((b, c) + (-u(a), v(a)) + \Delta\big) &= \phi\big((b - u(a), c + v(a)) + \Delta\big) \\
&= s'(b - u(a)) + r'(c + v(a)) \qquad \text{(E4)} \\
&= s'(b) + r'(c) \\
&= \phi\big((b, c) + \Delta\big). \qquad \text{(E5)}
\end{aligned}$$

$$\begin{aligned}
\phi\big((b + b', c + c') + \Delta\big) &= s'(b + b') + r'(c + c') \qquad \text{(E6)} \\
&= s'(b) + r'(c) + s'(b') + r'(c') \\
&= \phi\big((b, c) + \Delta\big) + \phi\big((b', c') + \Delta\big). \qquad \text{(E7)}
\end{aligned}$$

The composition of $\phi$ with the natural mapping $(a, b) \mapsto (a, b) + \Delta$ is continuous, hence $\phi$ is continuous. Since $\phi(s(b)) = \phi((b, 0) + \Delta) = s'(b)$ and $\phi(r(c)) = \phi((0, c) + \Delta) = r'(c)$, $\phi$ makes (E3) commutative.

To see that $\phi$ is unique suppose that there exists another continuous homomorphism $\phi' : PO \to G$ such that $\phi' \circ s = s'$ and $\phi' \circ r = r'$. Then

$$\begin{aligned}
\phi'\big((b, c) + \Delta\big) &= \phi'\big((b, 0) + \Delta\big) + \phi'\big((0, c) + \Delta\big) = \phi'\big(s(b)\big) + \phi'\big(r(c)\big) \\
&= s'(b) + r'(c) = \phi\big((b, c) + \Delta\big).
\end{aligned}$$

Thus $\phi = \phi'$. $\square$

*(ii) Suppose that there exists another triple $(PO', r' : B \to PO', s' : C \to PO')$ satisfying the universal property described in (i). Then $PO$ and $PO'$ are canonically isomorphic i.e. there exists an isomorphism $PO \to PO'$, making the corresponding diagram commutative.*

*Proof.* This is a consequence of (i). □

**(2.2.2) Proposition-definition.** *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups, $Y$ a topological abelian group and $t : H \to Y$, a continuous homomorphism. If $(PO, r, s)$ is the push-out of $\imath$ and $t$, there is an extension of topological abelian groups $tE$, making the diagram*

$$\begin{array}{rccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \downarrow t & & \downarrow s & & \| & & \\ tE: & 0 & \longrightarrow & Y & \xrightarrow[r]{} & PO & \dashrightarrow_{p} & G & \longrightarrow & 0 \end{array} \tag{E8}$$

*commutative. The extension $tE$ is called the* push-out extension *of $E$ and $t$.*

*Proof.* Consider the map $p : PO = (X \times Y)/\Delta \to G$; $(x, y) + \Delta \mapsto \pi(x)$, where $\Delta = \{(-\imath(h), t(h)) : h \in H\}$. Notice that $p$ is well-defined because

$$p\big((x, y) + (-\imath(h), t(h)) + \Delta\big) = \pi\big(x - \imath(h)\big) = \pi(x) = p\big((x, y) + \Delta\big),$$

furthermore, by construction $p$ makes (E8) commutative. The continuity of $p$ follows from the fact that its composition with the mapping $(x, y) \mapsto (x, y) + \Delta$ is continuous. To see that $p$ is open notice that for every $V \in \mathcal{N}_0(PO)$, $p(V) \supset p(s(s^{-1}(V))) = \pi(s^{-1}(V))$. This inclusion combined with the fact that $\pi$ is open and $s$ is continuous gives us that $p(V) \in \mathcal{N}_0(G)$.

The surjectivity of $\pi$ trivially implies that $p$ is onto. Furthermore, $r : Y \to PO$; $y \mapsto (0, y) + \Delta$ is one-to-one because if $(0, y) \in \Delta$, there exist $h \in H$ with $0 = -\imath(h)$ and $y = t(h)$ which implies that $y = 0$ (by the injectivity of $\imath$). Since

$$\begin{aligned} \ker p &= \big\{(x, y) + \Delta : \pi(x) = 0\big\} = \big\{(x, y) + \Delta : \exists h \text{ such that } x = -\imath(h)\big\} \\ &= \big\{(0, y) + \Delta : y \in Y\big\} = r(Y), \end{aligned}$$

$tE$ is a short exact sequence.

It only remains to check that $r : Y \to r(Y)$ is open. Pick $V \in \mathcal{N}_0(Y)$. We have to show that:

$$\exists U' \in \mathcal{N}_0(X), \exists V' \in \mathcal{N}_0(Y) : \ r(V) \supset (U' \times V' + \Delta) \cap r(Y). \tag{E9}$$

Since $r(y) = (0, y) + \Delta$, (E9) is equivalent to:

$$\begin{aligned} &\exists U' \in \mathcal{N}_0(X), \exists V' \in \mathcal{N}_0(Y) : \\ &\qquad (0, y) + \Delta \in (U' \times V' + \Delta) \Longrightarrow (0, y) + \Delta \in (\{0\} \times V + \Delta). \end{aligned} \tag{E10}$$

Since $r$ is injective, (E10) is actually equivalent to:

$$\exists U' \in \mathcal{N}_0(X), \exists V' \in \mathcal{N}_0(Y):$$
$$(0,y) + \Delta \in (U' \times V' + \Delta) \Longrightarrow y \in V. \tag{E11}$$

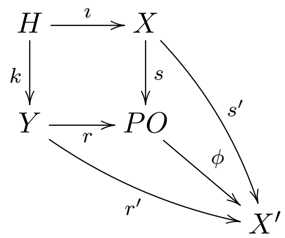
Taking a symmetric $V' \in \mathcal{N}_0(Y)$ with $V' + V' \subset V$ and $U' = -\imath(W)$ with $t(W) \subset V'$, $W \in \mathcal{N}_0(H)$, we obtain (E11), which proves that $r$ is open. ■

**(2.2.3) Proposition.** *Let $E: 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ and $E': 0 \to Y \overset{r'}{\to} X' \overset{p'}{\to} G \to 0$ be extensions of topological abelian groups and let $k: H \to Y$, $s': X \to X'$ be continuous homomorphism. Assume that the diagram (E12) is commutative*

$$\begin{array}{llllllllll} E: & 0 & \longrightarrow & H & \overset{i}{\longrightarrow} & X & \overset{\pi}{\longrightarrow} & G & \longrightarrow & 0 \\ & & & \downarrow k & & \downarrow s' & & \| & & \\ E': & 0 & \longrightarrow & Y & \underset{r'}{\longrightarrow} & X' & \underset{p'}{\longrightarrow} & G & \longrightarrow & 0 \end{array} \tag{E12}$$

*Then $E'$ is equivalent to the push-out extension $kE$ defined in (2.2.2).*

*Proof.* Consider the push-out triple $(PO, r, s)$ of $\imath$ and $k$ as in (2.2.1). In view of the proof of (2.2.1.i), the continuous homomorphism $\phi : PO = (X \times Y)/\Delta \to X'$; $(x,y) \mapsto s'(x) + r'(y)$ makes the following diagram commutative:



(E13)

Let us see that $\phi$ witnesses the equivalence between $E'$ and $kE$, in other words, that $\phi$ makes the squares in (E14) commutative.

$$\begin{array}{llllllllll} kE: & 0 & \longrightarrow & Y & \overset{r}{\longrightarrow} & PO & \overset{p}{\longrightarrow} & G & \longrightarrow & 0 \\ & & & \| & & \downarrow \phi & & \| & & \\ E': & 0 & \longrightarrow & Y & \underset{r'}{\longrightarrow} & X' & \underset{p'}{\longrightarrow} & G & \longrightarrow & 0 \end{array} \tag{E14}$$

Indeed, the commutativity of left square in (E14) is a trivial consequence of (E13). Furthermore, using the exactness of $E'$ and the commutativity of (E12)

$$p'\Big(\phi\big((x,y)+\Delta\big)\Big) = p'\big(r'(y) + s'(x)\big) = 0 + p'\big(s'(x)\big) = \pi(x) = p\big((x,y)+\Delta\big).$$

Since we are working in the class of Hausdorff topological abelian groups, it is necessary to check that when we consider the push-out extension $kE$ we stay in the class of Hausdorff topological abelian groups. This is a consequence of the following fact:

*Fact. Let $A, B$ and $C$ be Hausdorff topological abelian groups. If $u : A \to B$ is an embedding of topological abelian groups with closed image then for every continuous homomorphism $v : A \to C$, the associated push-out $PO = (B \times C)/\Delta = (B \times C)/\{(-u(a), v(a)) : a \in A\}$ is Hausdorff.*

Using that $u$ is an embedding, we can write

$$\begin{aligned}\Delta &= \{(-u(a), v(a)) : a \in A\} \\ &= \big\{(b, -v(u^{-1}(b))) : b \in u(A)\big\} \\ &= \mathrm{Graph}\big((-v) \circ (u^{-1}_{|u(A)})\big).\end{aligned}$$

Since $(-v)\circ(u^{-1}_{|u(A)}) : u(A) \to C$ is continuous and $B$ is Hausdorff, $\mathrm{Graph}\big((-v)\circ(u^{-1}_{|u(A)})\big)$ is closed in $u(A) \times C$ (see [Eng89, Cor. 2.3.22]), which is closed in $B \times C$. Accordingly, $\Delta$ is closed in $B \times C$ and $PO$, being the quotient of a Hausdorff group by a closed subspace, it is Hausdorff.

■

**(2.2.4) Corollary.** *Consider extensions of topological abelian groups $E : 0 \to H \to X \to G \to 0$, $E' : 0 \to H \to X' \to G \to 0$ and a continuous homomorphism $k : H \to Y$. If $E \equiv E'$ then $kE \equiv kE'$.*

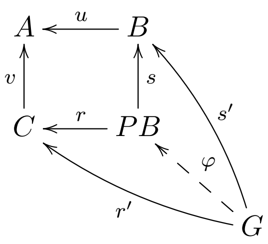
**(2.2.5) Pull-back square.** Let $A$, $B$ and $C$ be topological abelian groups and $u : B \to A$, $v : C \to A$ continuous homomorphisms. Define the topological abelian group $PB = \{(b, c) \in B \times C : u(b) = v(c)\} \leq B \times C$, and $r : PB \to C$; $(b, c) \mapsto c$, $s : PB \to B$; $(b, c) \mapsto b$. The following diagram is commutative:

$$\begin{array}{ccc} PB & \xrightarrow{s} & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & A \end{array}$$

The triple $(PB, r, s)$ is called the *pull-back* of $u$ and $v$ and it has the following universal property:

*(i) Given a topological abelian group $G$ and continuous homomorphisms $r' : G \to C$, $s' : G \to B$ with $v \circ r' = u \circ s'$, there is a unique continuous*

*homomorphism* $\varphi : G \to PB$ *making* (E15) *commutative.*




(E15)

*Proof.* The continuous homomorphism $\varphi : G \to PB$; $x \mapsto (s'(x), r'(x))$ makes (E15) commutative. To prove the uniqueness of $\varphi$ suppose that $\varphi' : G \to PB$ is another continuous homomorphism such that $s \circ \varphi' = s'$ and $r \circ \varphi' = r'$. Write $\varphi'(x) = (\varphi_1'(x), \varphi_2'(x))\ \forall x \in G$. $\varphi_1'(x) = s \circ \varphi'(x) = s'(x)$ and $\varphi_2'(x) = r \circ \varphi'(x) = r(x)$, hence $\varphi' = \varphi$. □

*(ii) Suppose that there exists another triple* $(PB', r' : PB' \to C, s' : PB' \to B)$ *satisfying the universal property described in (i). Then* $PB$ *and* $PB'$ *are canonically isomorphic i.e. there exists an isomorphism* $PB' \to PB$, *making the corresponding diagram commutative.*

*Proof.* This is a trivial consequence of (i). □

**(2.2.6) Proposition-definition.** *Let* $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *be an extension of topological abelian groups and let* $t : Y \longrightarrow G$ *be a continuous homomorphism. Let* $(PB, r, s)$ *be the pull-back of* $\pi$ *and* $t$. *There exists an extension of topological abelian groups* $Et$ *completing the commutative diagram* (E16)

$$\begin{array}{lccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \| & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \\ Et: & :0 & \longrightarrow & H & \overset{I}{\dashrightarrow} & PB & \xrightarrow{r} & Y & \longrightarrow & 0 \end{array} \qquad \text{(E16)}$$

*The extension* $Et$ *is called the* pull-back *extension of* $E$ *and* $t$.

*Proof.* The (well-defined) continuous homomorphism

$$\begin{array}{rlcl} I: & H & \longrightarrow & PB = \{(x, y) \in X \times Y : \pi(x) = t(y)\} \\ & h & \longmapsto & (\imath(h), 0) \end{array}$$

makes (E16) commutative.

Let us see that $Et$ is an extension of topological abelian groups. Since $\imath$ is an embedding and $I(H) = \imath(H) \times 0$, $I$ is also an embedding. Moreover, as $\ker r = \{(x, 0) \in PB\} = \{(x, 0) : \pi(x) = 0, x \in X\} = \imath(H) \times \{0\}$, the sequence $Et$ is exact. The surjectivity of $\pi$ implies that for every $y \in Y$, there exists $x \in X$ such that $\pi(x) = t(y)$, which means that $r$ is also onto. To check that $r$ is open, notice that for every $U \in \mathcal{N}_0(X), V \in \mathcal{N}_0(Y)$,

$$r\big((U \times V) \cap PB\big) \supset t^{-1}(\pi(U)) \cap V.$$

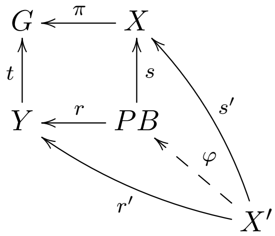
Since $\pi$ is open, $r((U \times V) \cap PB) \in \mathcal{N}_0(Y)$ and this gives us that $r$ is open. ∎

**(2.2.7) Proposition.** *Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ and $E' : 0 \to H \overset{I'}{\to} X' \overset{r'}{\to} Y \to 0$ be extensions of topological abelian groups and let $t : Y \to G$, $s' : X' \to X$ be two continuous homomorphisms. Assume that the diagram (E17) is commutative*

$$\begin{array}{lccccccccc} E: & 0 & \longrightarrow & H & \overset{\imath}{\longrightarrow} & X & \overset{\pi}{\longrightarrow} & G & \longrightarrow & 0 \\ & & & \| & & \uparrow s' & & \uparrow t & & \\ E': & :0 & \longrightarrow & H & \underset{I'}{\longrightarrow} & X' & \underset{r'}{\longrightarrow} & Y & \longrightarrow & 0 \end{array} \qquad \text{(E17)}$$

*Then $E'$ is equivalent to the pull-back extension $Et$ defined in (2.2.6).*

*Proof.* Let $(PB, r, s)$ the pull-back of $\pi$ and $t$. In view of (2.2.5.i), the continuous homomorphism $\varphi : X' \mapsto PB;\ x \mapsto (s'(x), r'(x))$ makes commutative the diagram




(E18)

It suffices to show that the diagram (E19) is commutative.

$$\begin{array}{lccccccccc} E': & :0 & \longrightarrow & H & \overset{I'}{\longrightarrow} & X' & \overset{r'}{\longrightarrow} & Y & \longrightarrow & 0 \\ & & & \| & & \downarrow \varphi & & \| & & \\ Et: & :0 & \longrightarrow & H & \underset{I}{\longrightarrow} & PB & \underset{r}{\longrightarrow} & Y & \longrightarrow & 0 \end{array} \qquad \text{(E19)}$$

The commutativity of the right square of (E19) follows from the commutativity of (E18). Using (E17)

$$\varphi\big(I'(h)\big) = \big(s'\big(I'(h)\big), r'\big(I'(h)\big)\big) = \big(\imath(h), 0\big) = I(h),$$

which completes the proof. ∎

**(2.2.8) Corollary.** *Consider two extensions of topological abelian groups* $E : 0 \to H \to X \to G \to 0$ *and* $E' : 0 \to H \to X' \to Y \to 0$ *and a continuous homomorphisms* $t : Y \to G$*. If* $E \equiv E'$ *then* $Et \equiv E't$*.*

**(2.2.9) Proposition.** *Let* $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *be an extension of topological abelian groups and let* $k : H \to H'$*,* $t : G' \to G$ *be continuous homomorphisms. Then the extensions* $k(Et)$ *and* $(kE)t$ *are equivalent.*

*Sketch of the proof.* Displaying the extensions $E$, $kE$, $(kE)t$, $Et$ and $k(Et)$ we obtain the following commutative diagram

$$
\begin{array}{lccccccccc}
(kE)t: & 0 & \longrightarrow & H' & \longrightarrow & PB_{(kE)t} & \longrightarrow & G' & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle t} & & \\
kE: & 0 & \longrightarrow & H' & \longrightarrow & PO_{kE} & \longrightarrow & G & \longrightarrow & 0 \\
 & & & {\scriptstyle k}\uparrow & & \uparrow & & \| & & \\
E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & {\scriptstyle t}\uparrow & & \\
Et: & 0 & \longrightarrow & H & \longrightarrow & PB_{Et} & \longrightarrow & G' & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle k} & & \downarrow & & \| & & \\
k(Et): & 0 & \longrightarrow & H' & \longrightarrow & PO_{k(Et)} & \longrightarrow & G' & \longrightarrow & 0
\end{array}
$$

where the unspecified arrows are the canonical mappings and

$$
\begin{aligned}
PO_{kE} &= (H' \times X)/\{(-k(h), \imath(h)) : h \in H\} = (H' \times X)/\Delta_1 \\
PB_{(kE)t} &= \{((h', x) + \Delta_1, g') : t(g') = \pi(x)\} \leq PO_{kE} \times G' \\
PB_{Et} &= \{(x, g') : t(g') = \pi(x)\} \leq X \times G' \\
PO_{k(Et)} &= (H' \times PB_{Et}))/\{(-k(h), (\imath(h), 0)) : h \in H\} = (H' \times PB_{Et})/\Delta_2
\end{aligned}
$$

A routine verification shows that the continuous homomorphism

$$
\begin{array}{ccc}
PB_{(kE)t} & \longrightarrow & PO_{k(Et)} \\
\big((h', x) + \Delta_1, g'\big) & \longmapsto & \big(h', (x, g')\big) + \Delta_2
\end{array}
$$

witnesses the equivalence of $(kE)t$ and $k(Et)$. ∎

**(2.2.10) Notes.** (2.2.2) and (2.2.6) were introduced in the category of topological abelian groups by Castillo in [Cas00]. (2.2.3) and (2.2.7) appear in [Bel] without proof; the argument shown here imitates the one for abelian groups ([ML95, lemmas 1.2, 1.3, 1.4 and 1.5 of Ch. III]).

# Chapter 3

# The Ext group in the category of topological abelian groups

We proceed now to define the Ext group in the context of topological abelian groups. The definition of this object will be a direct translation of the definition of Ext commonly used in homological algebra (1.1.5). In sections §3.2 to §3.5 we will study the properties of this group and we will apply them to investigate Problem 2.

The results of this chapter continue the work of Fulp and Griffith ([FG71a], [FG71b]) and Sahleh and Alijani ([SA14a], [SA14b]) which was focused on the class of locally compact abelian groups.

## §3.1 Addition of extensions

**(3.1.1) The Ext group.** Let $G, H$ be topological abelian groups. We denote by $\mathrm{Ext}(G, H)$ the set of all equivalence classes of extensions of topological abelian groups of the form $0 \to H \to X \to G \to 0$, with the equivalence relation defined in (2.1.1).

Consider the natural maps $\Delta_G : G \longrightarrow G \times G;\ g \mapsto (g, g)$ and $\nabla_H : H \times H \longrightarrow H; (h, h') \mapsto h + h'$. Notice that given two extensions of topological abelian groups $E_1 : 0 \to H \to X_1 \to G \to 0$ and $E_2 : 0 \to H \to X_2 \to G \to 0$, its product $E_1 \times E_2$ (defined as in (1.1.5)) is an extension of topological abelian groups. Using the terminology of lemmas (2.2.6) and (2.2.2), we will define the addition of the equivalence classes $[E_1]$ and $[E_2]$ as

$$[E_1] + [E_2] = \Big[\big(\nabla_H(E_1 \times E_2)\big)\Delta_G\Big] = \Big[\nabla_H\big((E_1 \times E_2)\Delta_G\big)\Big]. \qquad \text{(E20)}$$

The second identity is a consequence of (2.2.9). This operation is called the *Baer sum* and although it is defined between equivalence classes, some-times we will write $E_1 + E_2 = \nabla_H(E_1 \times E_2)\Delta_G$ as a way to shorten the expressions.

*(i)* $\mathrm{Ext}(G,H)$ *endowed with the Baer sum (E20) is an abelian group. Furthermore,*

$$k(E_1+E_2) \equiv kE_1 + kE_2, \quad (E_1+E_2)t \equiv E_2t + E_2t, \tag{E21}$$
$$(k_1+k_2)E \equiv k_1E + k_2E, \quad E(t_1+t_2) \equiv Et_1 + Et_2, \tag{E22}$$

*for every* $[E],[E_1],[E_2] \in \mathrm{Ext}(G,H)$ *and continuous homomorphisms* $t, t_1, t_2$ *from a topological abelian group* $Y_1$ *to* $G$ *and continuous homomorphisms* $k, k_1, k_2$ *from* $H$ *to another topological abelian group* $Y_2$.

*Proof.* The proof given by Mac Lane in [ML95, Th. 2.1 of Ch. III] for extensions of modules is valid in this context with the obvious replacements. Nevertheless, we will include the proof for the sake of completeness.

We will start proving (E21) and (E22). The following identities are immediate from the definitions of $\Delta_G$ and $\nabla_H$,

$$k \circ \nabla_H = \nabla_{Y_2} \circ (k \times k), \tag{E23}$$
$$\Delta_G \circ t = (t \times t) \circ \Delta_{Y_1}. \tag{E24}$$

In view of (E23),

$$\begin{aligned} k(E_1+E_2) &\equiv k(\nabla_H(E_1\times E_2)\Delta_G) \equiv (k\circ\nabla_H)(E_1\times E_2)\Delta_G \\ &\equiv \Big(\nabla_{Y_2}\circ(k\times k)\Big)(E_1\times E_2)\Delta_G \equiv \nabla_{Y_2}(kE_1\times kE_2)\Delta_G \\ &\equiv kE_1 + kE_2. \end{aligned}$$

Notice that the third and fourth equivalences are consequences of the uniqueness of the pull-back and push-out extensions ((2.2.7) and (2.2.3)). The second part of (E21) can be proven with an analogous argument using (E24) instead of (E23). To check (E22) notice that

$$E\nabla_G \equiv \nabla_H(E\times E), \tag{E25}$$
$$\Delta_H E \equiv (E\times E)\Delta_G. \tag{E26}$$

Using (E25),

$$\begin{aligned} Et_1 + Et_2 &\equiv \nabla_H(Et_1\times Et_2)\Delta_{Y_1} \equiv \nabla_H(E\times E)\Big((t_1\times t_2)\circ\Delta_{Y_1}\Big) \\ &\equiv E\Big(\nabla_H\circ(t_1\times t_2)\circ\Delta_{Y_1}\Big) \equiv E(t_1+t_2). \end{aligned}$$

The same argument (using (E26)) proves the second part of (E22).

From the identities

$$\nabla_H\circ(\nabla_H\times \mathrm{Id}_H) = \nabla_H\circ(\mathrm{Id}_H\times\nabla_H),$$
$$(\Delta_G\times\mathrm{Id}_G)\circ\Delta_G = (\mathrm{Id}_G\times\Delta_G)\circ\Delta_G,$$

we deduce that

$$\begin{aligned}
E_1 + (E_2 + E_3) &\equiv \nabla_H\Big(E_1 \times \big(\nabla_H(E_2 \times E_3)\Delta_G\big)\Big)\Delta_G \\
&\equiv \nabla_H\big(E_1 \times \nabla_H(E_2 \times E_3)\big)(\mathrm{Id}_G \times \Delta_G) \circ \Delta_G \\
&\equiv \nabla_H \circ (\mathrm{Id}_H \times \nabla_H)\big(E_1 \times E_2 \times E_3\big)(\mathrm{Id}_G \times \Delta_G) \circ \Delta_G \\
&\equiv \nabla_H \circ (\nabla_H \times \mathrm{Id}_H)\big(E_1 \times E_2 \times E_3\big)(\Delta_G \times \mathrm{Id}_G) \circ \Delta_G \\
&\equiv \nabla_H\big(\nabla_H(E_1 \times E_2) \times E_3\big)(\Delta_G \times \mathrm{Id}_G) \circ \Delta_G \\
&\equiv \nabla_H\Big(\big(\nabla_H(E_1 \times E_2)\Delta_G\big) \times E_3\Big)\Delta_G \\
&\equiv (E_1 + E_2) + E_3.
\end{aligned}$$

which gives us the associative law.

Consider now the natural isomorphisms

$$\begin{array}{rccc}
\phi_H: & H \times H & \longrightarrow & H \times H \\
& (h_1, h_2) & \longmapsto & (h_2, h_1)
\end{array}
\qquad
\begin{array}{rccc}
\phi_G: & G \times G & \longrightarrow & G \times G \\
& (g_1, g_2) & \longmapsto & (g_2, g_1)
\end{array}$$

An easy verification shows that $\nabla_H \circ \phi_H = \nabla_H$, $\phi_G \circ \Delta_G = \Delta_G$ and $\phi_H(E_1 \times E_2) \equiv (E_2 \times E_1)\phi_G$. The following equivalences prove the commutative law

$$\begin{aligned}
E_1 + E_2 &\equiv \nabla_H(E_1 \times E_2)\Delta_G \equiv \nabla_H \circ \phi_H(E_1 \times E_2)\Delta_G \\
&\equiv \nabla_H(E_2 \times E_1)\phi_G \circ \Delta_G \equiv \nabla_H(E_2 \times E_1)\Delta_G \\
&\equiv E_2 + E_1.
\end{aligned}$$

To show that the trivial extension $E_0 : 0 \to H \stackrel{\imath_H}{\to} H \times G \stackrel{\pi_G}{\to} G \to 0$ acts as the zero for the Baer sum, first observe that given an extension of topological abelian groups $E : 0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$, we can always construct the commutative diagram

$$\begin{array}{llcccccccc}
E: & :0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 0_H} & & \downarrow{\scriptstyle v} & & \| & & \\
E_0: & :0 & \longrightarrow & H & \xrightarrow[\imath_H]{} & H \times G & \xrightarrow[\pi_G]{} & G & \longrightarrow & 0
\end{array}$$

taking $v(x) = (0, \pi(x))$. According to (2.2.3) $E_0$ is equivalent to $0_H E$. In virtue of (E22)

$$E + E_0 \equiv E + 0_H E \equiv (\mathrm{Id}_H + 0_H)E \equiv E.$$

Similarly,

$$E + (-\mathrm{Id}_H)E \equiv (\mathrm{Id}_H)E + (-\mathrm{Id}_H)E \equiv (\mathrm{Id}_H - \mathrm{Id}_H)E \equiv 0_H E \equiv E_0,$$

hence $(-\mathrm{Id}_H)E$ acts as the additive inverse of $E$ under the Baer sum. This completes the proof of (i). $\square$

From (E21) one easily obtains the following result:
*(ii) Let $G$, $H$, $G'$ and $H'$ be topological abelian groups. Suppose that $t : G' \to G$ and $k : H \to H'$ are continuous homomorphisms; then the following maps are homomorphisms of abelian groups:*

$$\begin{array}{ccc} \mathrm{Ext}(G,H) & \longrightarrow & \mathrm{Ext}(G',H) \\ [E] & \longmapsto & [Et] \end{array} \qquad \begin{array}{ccc} \mathrm{Ext}(G,H) & \longrightarrow & \mathrm{Ext}(G,H') \\ [E] & \longmapsto & [kE] \end{array}$$

**(3.1.2) Ext in $\mathcal{L}$.** Fulp and Griffith defined in [FG71a] the group Ext in the class $\mathcal{L}$ using the Baer sum as in (3.1.1). Notice that since local compactness is a three space property, by (2.1.4) both definitions of Ext coincide when we take groups in $\mathcal{L}$.

## §3.2 Ext and dense subgroups

**(3.2.1) Lemma.** *If $G$ is a Raĭkov-complete topological abelian group and $H \leq G$ is a closed Čech-complete subgroup of $G$ then $G/H$ is also Raĭkov-complete.*

*Proof.* This is [RD81b, 11.18]. ■

**(3.2.2) Proposition.** *Let $G$ be a topological abelian group and $H \leq G$ a closed subgroup of $G$. If the Raĭkov completion of $H$ is Čech-complete, then the canonical map $\varphi\colon G/H \to \varrho G/\varrho H$ is a dense embedding which extends to a topological isomorphism of $\varrho(G/H)$ onto $\varrho G/\varrho H$.*

*Proof.* Let $\pi_H\colon G \to G/H$ and $\pi_{\varrho H}\colon \varrho G \to \varrho G/\varrho H$ be the canonical projections. Note that $\varphi(\pi_H(g)) = \pi_{\varrho H}(g)$ for every $g \in G$. It is clear that $\varphi$ is a continuous monomorphism. That $\varphi(G/H) = \pi_{\varrho H}(G)$ is dense in $\varrho G/\varrho H$ follows from the fact that $G$ is dense in $\varrho G$ and $\pi_{\varrho H}$ is a quotient map.

Let us see that $\varphi$ is relatively open. Fix a closed neighborhood $U$ of $e$ in $G$. Let us show that for every symmetric neighborhood $V$ of $e$ in $G$ with $V + V \subseteq U$, we have $\varphi(\pi_H(U)) \supset \varphi(G/H) \cap \pi_{\varrho H}(\overline{V})$ or, equivalently, $\pi_{\varrho H}(U) \supset \pi_{\varrho H}(G) \cap \pi_{\varrho H}(\overline{V})$ where the closure is taken in $\varrho G$. Fix $z \in \overline{V}$ and $g \in G$ with $z - g \in \varrho H$. Fix $h \in H \cap (z - g + \overline{V})$. Then $u = h + g$ satisfies $u \in (\overline{V} + \overline{V}) \cap G \subseteq \overline{U} \cap G = U$ and $z - u \in \varrho H$.

The group $\varrho G/\varrho H$ is complete by (3.2.1). Hence $\varrho\varphi\colon \varrho(G/H) \to \varrho G/\varrho H$ is a topological isomorphism. ■

**(3.2.3) Proposition.** *Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ be an extension of topological abelian groups. Suppose that the Raĭkov completion of $H$ is a Čech-complete group. Then the sequence $\varrho E : 0 \to \varrho H \overset{\varrho\imath}{\to} \varrho X \overset{\varrho\pi}{\to} \varrho G \to 0$ is an extension of topological abelian groups.*

*Proof.* Consider the topological isomorphism $\alpha\colon G \to X/\imath(H)$ defined by $\alpha(g) = \pi(g) + \imath(H)$. Note that the completion $\varrho\alpha\colon \varrho G \to \varrho(X/\imath(H))$ of $\alpha$ is a topological isomorphism, too.

Regarded as a subgroup of $\varrho X$, $\varrho(\imath(H))$ coincides with $\varrho\imath(\varrho H)$. Let $\varphi\colon \varrho(X/\imath(H)) \to \varrho X/\varrho\imath(\varrho H)$ be the topological isomorphism of (3.2.2). It is easy to check that the diagram

$$\begin{array}{lccccccccc}
\varrho E: & 0 & \longrightarrow & \varrho H & \xrightarrow{\varrho\imath} & \varrho X & \xrightarrow{\varrho\pi} & \varrho G & \longrightarrow & 0 \\
 & & & \big\downarrow{\scriptstyle \widetilde{\varrho\imath}} & & \big\| & & \big\downarrow{\scriptstyle \varphi\circ\varrho\alpha} & & \\
E': & 0 & \longrightarrow & \varrho\imath(\varrho H) & \longrightarrow & \varrho X & \longrightarrow & \varrho X/\varrho\imath(\varrho H) & \longrightarrow & 0
\end{array}$$

is commutative, where $E'$ is the canonical extension and $\widetilde{\varrho\imath}$ is the corestriction of $\varrho\imath$. Since the downward maps are topological isomorphisms and $E'$ is an extension of topological abelian groups, $\varrho E$ is an extension too. ■

**(3.2.4) Theorem.** *Let $G, H$ be topological abelian groups. If $H$ is Čech-complete then* $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(\varrho G, H)$.

*Proof.* Let $\mathcal{I} : G \hookrightarrow \varrho G$ be the canonical inclusion. According to (3.1.1) the map $\phi : \mathrm{Ext}(\varrho G, H) \to \mathrm{Ext}(G, H)$; $[E] \mapsto [E\mathcal{I}]$ is a homomorphism of abelian groups. Let us see that $\phi$ is an isomorphism.

To prove that $\phi$ is one-to-one, pick an extension of topological abelian groups $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} \varrho G \to 0$ and suppose that $E\mathcal{I}$ splits. The sequence $E' : 0 \to H \xrightarrow{\imath} \pi^{-1}(G) \xrightarrow{\pi_{|\pi^{-1}(G)}} G \to 0$ is also an extension of topological abelian groups. Furthermore, the following diagram is commutative:

$$\begin{array}{lccccccccc}
E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & \varrho G & \longrightarrow & 0 \\
 & & & \big\| & & \big\uparrow & & \big\uparrow{\scriptstyle \mathcal{I}} & & \\
E': & 0 & \longrightarrow & H & \xrightarrow{\imath} & \pi^{-1}(G) & \xrightarrow[\pi_{|\pi^{-1}(G)}]{} & G & \longrightarrow & 0
\end{array}$$

According to (2.2.7), $E'$ must be equivalent to $E\mathcal{I}$. Then $E'$ splits and applying (2.1.7) we find a continuous homomorphism $P : \pi^{-1}(G) \to H$ such that $P \circ \imath = \mathrm{Id}_H$. Since $G$ is dense in $\varrho G$, it is clear that $\pi^{-1}(G)$ is dense in $X$. Let $R : X \to H$ be an extension of $P$ to $X$ defined canonically. $R$ is a continuous homomorphism satisfying $R \circ \imath = \mathrm{Id}_H$, hence by (2.1.7), $E$ splits.

To check that $\phi$ is onto, choose an extension of topological abelian groups $\mathcal{E} : 0 \to H \xrightarrow{I} Y \xrightarrow{p} G \to 0$. By (3.2.3) the sequence $\varrho\mathcal{E} : 0 \to H \xrightarrow{\varrho I} \varrho Y \xrightarrow{\varrho p} \varrho G \to 0$ is an extension of topological abelian groups, which gives us the following commutative diagram:

$$\begin{array}{lccccccccc} \varrho\mathcal{E}: & 0 & \longrightarrow & H & \xrightarrow{\varrho I} & \varrho Y & \xrightarrow{\varrho p} & \varrho G & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow \mathcal{I} & & \\ \mathcal{E}: & 0 & \longrightarrow & H & \xrightarrow{I} & Y & \xrightarrow{p} & G & \longrightarrow & 0 \end{array}$$

In virtue of (2.2.7), $\mathcal{E}$ must be equivalent to $(\varrho\mathcal{E})\mathcal{I}$ i.e. $\phi([\varrho\mathcal{E}]) = [\mathcal{E}]$. ■

**(3.2.5) Theorem.** *Let $G$ be a topological abelian group and let $H$ be a Čech-complete topological abelian group. Suppose that $D$ is a dense subgroup of $G$. Then $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(D, H)$.*

*Proof.* If $D$ is dense in $G$, then $\varrho D = \varrho G$ and by (3.2.4)

$$\mathrm{Ext}(D, H) \cong \mathrm{Ext}(\varrho D, H) = \mathrm{Ext}(\varrho G, H) \cong \mathrm{Ext}(G, H).$$

■

**(3.2.6) Applications to $\mathcal{L}$.** In theorems 3.5 and 3.6 of [FG71b] the authors study situations in which $\mathrm{Ext}(G, X) = 0$ for a fixed $G \in \mathcal{L}$ and $X$ varying in a subclass of $\mathcal{L}$.

They prove that for every group $G \in \mathcal{L}$:

(a) $\mathrm{Ext}(G, X) = 0$ *for all totally disconnected* $X \in \mathcal{L}$ *if and only if* $G \cong (\mathbb{Z}^{(k)})_d \times \mathbb{R}^n$ *where* $n < \omega$, $k$ *is an ordinal number and* $(\mathbb{Z}^{(k)})_d$ *stands for the direct sum* $\mathbb{Z}^{(k)}$ *endowed with the discrete topology* [FG71b, Th. 3.5].

(b) $\mathrm{Ext}(G, C) = 0$ *for all connected* $C \in \mathcal{L}$ *if and only if* $G \cong \mathbb{R}^n \times M$ *where* $n < \omega$ *and* $M$ *contains a compact open subgroup having a co-torsion Pontryagin dual* [FG71b, Th. 3.6].

Applying (3.2.4) we have:

*(a') If $G$ is locally precompact,* $\mathrm{Ext}(G, X) = 0$ *for all totally disconnected $X \in \mathcal{L}$ if and only if $\varrho G = (\mathbb{Z}^{(k)})_d \times \mathbb{R}^n$ for some $n < \omega$ and an arbitrary ordinal number $\kappa$.*

*Proof.* Suppose that a locally precompact abelian group $G$ has the property that $\mathrm{Ext}(G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. Since every group in $\mathcal{L}$ is Čech-complete, by (3.2.4), $\mathrm{Ext}(\varrho G, X) = \mathrm{Ext}(G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. By (a), $\varrho G = (\mathbb{Z}^{(k)})_d \times \mathbb{R}^n$ for some $n < \omega$ and $\alpha$ an ordinal number. Conversely if $\varrho G = (\mathbb{Z}^{(k)})_d \times \mathbb{R}^n$, again by (a), $\mathrm{Ext}(\varrho G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. Invoking (3.2.4) we obtain that $\mathrm{Ext}(G, X) = \mathrm{Ext}(\varrho G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. □

Using the same argument with (b) we obtain:

*(b') If $G$ is locally precompact,* $\mathrm{Ext}(G,X)=0$ *for all connected* $X\in\mathcal{L}$ *if and only if* $\varrho G=\mathbb{R}^n\times G'$ *where* $n<\omega$ *and* $G'$ *contains a compact open subgroup having a co-torsion dual.*

## §3.3 Ext and open subgroups

**(3.3.1) Lemma.** *Let $A$ be an open subgroup of a topological group $G$ and suppose that an extension of topological abelian groups* $E:0\to H\overset{\imath}{\to}X\overset{\pi}{\to}A\to 0$ *splits algebraically. Then there exists a group topology $\tau$ on $H\times G$ and an embedding $f:X\to(H\times G,\tau)$ making commutative the diagram*

$$\begin{array}{cccccccccc} \overline{E}: & 0 & \longrightarrow & H & \xrightarrow{\imath_\tau} & (H\times G,\tau) & \xrightarrow{\pi_\tau} & G & \longrightarrow & 0 \\ & & & \| & & f\uparrow & & \uparrow & & \\ E & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array}$$

*where $\imath_\tau$ and $\pi_\tau$ are the canonical mappings and $\overline{E}$ is an extension of topological abelian groups.*

*Proof.* Since $E$ splits algebraically there exists a group topology $\tau'$ on $H\times A$ such that $E$ is equivalent to the extension of topological abelian groups $\mathcal{E}:0\to H\overset{\imath_{\tau'}}{\to}(H\times A,\tau')\overset{\pi_{\tau'}}{\to}A\to 0$ where $\imath_{\tau'}$ and $\pi_{\tau'}$ are respectively the canonical inclusion and the canonical projection (2.1.5.ii). Let $T$ be the topological isomorphism that makes (E27) commutative.

$$\begin{array}{cccccccccc} \mathcal{E}: & 0 & \longrightarrow & H & \xrightarrow{\imath_{\tau'}} & (H\times A,\tau') & \xrightarrow{\pi_{\tau'}} & A & \longrightarrow & 0 \\ & & & \| & & T\uparrow & & \| & & \\ E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array} \qquad \text{(E27)}$$

Now, consider the group topology $\tau$ on $H\times G$ obtained by declaring $(H\times A,\tau')$ an open subgroup. An easy verification shows that $\imath_\tau:H\to(H\times G,\tau)$; $h\mapsto(h,0)$ and $\pi_\tau:(H\times G,\tau)\to G$; $(h,g)\mapsto g$ give us an extension of topological abelian groups $\overline{E}:0\to H\overset{\imath_\tau}{\to}(H\times G,\tau)\overset{\pi_\tau}{\to}G\to 0$. Combining the commutative diagram:

$$\begin{array}{cccccccccc} \overline{E}: & 0 & \longrightarrow & H & \xrightarrow{\imath_\tau} & (H\times G,\tau) & \xrightarrow{\pi_\tau} & G & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow & & \\ \mathcal{E} & 0 & \longrightarrow & H & \xrightarrow{\imath_{\tau'}} & (H\times A,\tau') & \xrightarrow{\pi_{\tau'}} & A & \longrightarrow & 0 \end{array}$$

with (E27) and defining $f$ as the composition of $T$ and the inclusion $(H\times A,\tau')\hookrightarrow(H\times G,\tau)$, we complete the proof. ■

**(3.3.2) Theorem.** *Let $G, H$ be topological abelian groups. Suppose that $H$ is divisible and that $A$ is an open subgroup of $G$, then* $\operatorname{Ext}(G, H) \cong \operatorname{Ext}(A, H)$*.*

*Proof.* We will use the same strategy as in (3.2.4). Consider the canonical inclusion $\mathcal{I} : A \to G$. According to (3.1.1.ii) the map $\phi : \operatorname{Ext}(G, H) \mapsto \operatorname{Ext}(A, H)$; $[E] \mapsto [E\mathcal{I}]$ is a homomorphism of abelian groups. Let us see that $\phi$ is an isomorphism.

We will start proving that $\phi$ is one-to-one. Pick an extension $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ and suppose that $E\mathcal{I}$ splits. Since $\imath(H) = \pi^{-1}(0) \leq \pi^{-1}(G)$, the sequence $E' : 0 \to H \xrightarrow{\imath} \pi^{-1}(A) \xrightarrow{\pi_{|\pi^{-1}(A)}} G \to 0$ is exact. The mapping $\pi_{|\pi^{-1}(A)}$ is open, hence $E'$ is an extension of topological abelian groups. Furthermore the following diagram is commutative:

$$\begin{array}{lccccccccc}
E: & 0 & \longrightarrow & H & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow{\scriptstyle \mathcal{I}} & & \\
E': & 0 & \longrightarrow & H & \xrightarrow{\imath} & \pi^{-1}(A) & \xrightarrow{\pi_{|\pi^{-1}(A)}} & A & \longrightarrow & 0
\end{array}$$

According to (2.2.7), $E'$ must be equivalent to $E\mathcal{I}$. Then $E'$ splits and applying (2.1.7) we find a continuous homomorphism $P : \pi^{-1}(A) \to H$ such that $P \circ \imath = \mathrm{Id}_H$. Since $H$ is divisible we can extend the homomorphism $P$ to a homomorphism $R : X \to H$. Since $\pi^{-1}(A)$ is open in $X$ and $R_{|\pi^{-1}(A)} = P$, $R$ is a continuous homomorphism. As $\imath(H) \leq \pi^{-1}(A)$, $R$ satisfies $R \circ \imath = P \circ \imath = \mathrm{Id}_H$ and by (2.1.7), $E$ splits.

To check that $\phi$ is onto, choose an extension of topological abelian groups $E : 0 \to H \xrightarrow{I} Y \xrightarrow{p} A \to 0$. Since $E$ splits algebraically, invoking (3.3.1) we obtain that there exists a group topology $\tau$ on $H \times G$ and a commutative diagram

$$\begin{array}{lccccccccc}
\overline{E}: & 0 & \longrightarrow & H & \xrightarrow{\imath_\tau} & (H \times G, \tau) & \xrightarrow{\pi_\tau} & G & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow{\scriptstyle \mathcal{I}} & & \\
E & 0 & \longrightarrow & H & \xrightarrow{I} & Y & \xrightarrow{p} & A & \longrightarrow & 0
\end{array}$$

where $\imath_\tau$ and $\pi_\tau$ are the canonical mappings and $\overline{E}$ is an extension of topological abelian groups. In virtue of (2.2.7), $E$ is equivalent to $(\overline{E})\mathcal{I}$, concluding that $\phi(\big[\overline{E}\big]) = \big[(\overline{E})\mathcal{I}\big] = [E]$. ■

**(3.3.3) Remark on (3.3.2).** Theorem (3.3.2) is not true in general if we do not suppose that $H$ is divisible. Indeed, take $\mathbb{Q}_d$ the group of rational numbers endowed with the discrete topology, according to [FG71a, Exercise 51.7] $\operatorname{Ext}(\mathbb{Q}_d, \mathbb{Z}) \cong \mathbb{Q}^\omega$. $\{0\}$ is trivially an open subgroup of $\mathbb{Q}_d$ but

$$\operatorname{Ext}(\{0\}, \mathbb{Z}) \cong 0 \ncong \mathbb{Q}^\omega \cong \operatorname{Ext}(\mathbb{Q}_d, \mathbb{Z}).$$

## §3.4 Ext, products and coproducts

**(3.4.1) Theorem.** *Let $G$ be a topological abelian group and let $\{H_\alpha : \alpha < \kappa\}$ be a family of topological abelian groups. Then* $\mathrm{Ext}(G, \prod_{\alpha<\kappa} H_\alpha) \cong \prod_{\alpha<\kappa} \mathrm{Ext}(G, H_\alpha)$.

*Proof.* Consider for every $\beta < \kappa$ the canonical projection $p_\beta : \prod_{\alpha<\kappa} H_\alpha \to H_\beta$.

Given an extension of topological abelian groups $E : 0 \to \prod_{\alpha<\kappa} H_\alpha \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$, take the push-out extension $p_\beta E : 0 \to H_\beta \xrightarrow{r_\beta} PO_\beta \xrightarrow{P_\beta} G \to 0$ and consider the commutative diagram (E28) as in (2.2.2).

$$\begin{array}{llcccccccccl} E: & 0 & \longrightarrow & \prod_{\alpha<\kappa} H_\alpha & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 & & \\ & & & \downarrow{\scriptstyle p_\beta} & & \downarrow{\scriptstyle s_\beta} & & \| & & & & \text{(E28)} \\ p_\beta E: & 0 & \longrightarrow & H_\beta & \xrightarrow{r_\beta} & PO_\beta & \xrightarrow{P_\beta} & G & \longrightarrow & 0 & & \end{array}$$

In virtue of (3.1.1) the map

$$\begin{array}{rccc} \phi: & \mathrm{Ext}(G, \prod_{\alpha<\kappa} H_\alpha) & \longrightarrow & \prod_{\alpha<\kappa} \mathrm{Ext}(G, H_\alpha) \\ & [E] & \longmapsto & ([p_\alpha E])_{\alpha<\kappa} \end{array}$$

is a homomorphism of abelian groups.

Let us check that $\phi$ is one-to-one. Take an extension of topological groups $E : 0 \to \prod_{\alpha<\kappa} H_\alpha \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$, and suppose that $p_\beta E : 0 \to H_\beta \xrightarrow{r_\beta} PO_\beta \xrightarrow{p_\beta} G \to 0$ splits for every $\beta < \kappa$. By (2.1.7) for every $\alpha < \kappa$ there exists a continuous homomorphism $t_\alpha : PO_\alpha \to H_\alpha$ with $t_\alpha \circ r_\alpha = \mathrm{Id}_{H_\alpha}$. Define the continuous homomorphism $T = (t_\alpha \circ s_\alpha)_{\alpha<\kappa} : X \to \prod_{\alpha<\kappa} H_\alpha$; $x \mapsto (t_\alpha \circ s_\alpha(x))_{\alpha<\kappa}$. By the commutativity of (E28), $T \circ \imath = \mathrm{Id}_{\prod H_\alpha}$, hence $E$ splits.

To see that $\phi$ is onto pick a family of extensions $\{E_\alpha : 0 \to H_\alpha \xrightarrow{\imath_\alpha} X_\alpha \xrightarrow{\pi_\alpha} G \to 0 : \alpha < \kappa\}$. Consider the extension $\mathcal{E} : 0 \to \prod_{\alpha<\kappa} H_\alpha \xrightarrow{\mathcal{I}} \mathcal{B} \xrightarrow{\mathcal{P}} G \to 0$, where

$$\mathcal{B} = \left\{ ((x_\alpha)_{\alpha<\kappa}, g) \in \prod_{\alpha<\kappa} X_\alpha \times G : \pi_\alpha(x_\alpha) = g \;\forall \alpha < \kappa \right\},$$

$\mathcal{I}((h_\alpha)_{\alpha<\kappa}) = ((\imath_\alpha(h_\alpha))_{\alpha<\kappa}, 0)$ and $\mathcal{P}((x_\alpha)_{\alpha<\kappa}, g) = g$. It is easy to check that $\mathcal{E}$ is an extension of topological abelian groups. For each $\beta < \kappa$ the continuous homomorphism $\mathcal{P}_\beta : \mathcal{B} \to X_\beta$; $((x_\alpha)_{\alpha<\kappa}, g) \mapsto x_\beta$ gives us the commutative diagram:

$$\begin{array}{lccccccccc} \mathcal{E}: & 0 & \longrightarrow & \prod_{\alpha<\kappa} H_\alpha & \xrightarrow{\mathcal{I}} & \mathcal{B} & \xrightarrow{\mathcal{P}} & G & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle p_\beta} & & \downarrow{\scriptstyle \mathcal{P}_\beta} & & \| & & \\ E_\beta: & 0 & \longrightarrow & H_\beta & \xrightarrow{\imath_\beta} & X_\beta & \xrightarrow{\pi_\beta} & G & \longrightarrow & 0 \end{array}$$

Consequently, by (2.2.3), $E_\beta$ is equivalent to the push-out sequence $p_\beta\mathcal{E}$. Therefore $\varphi([\mathcal{E}]) = ([E_\alpha])_{\alpha<\kappa}$, which concludes the proof. ■

**(3.4.2) Applications to $\mathcal{L}$.** In [SA14a] and [SA14b] the authors prove that for every $G \in \mathcal{L}$:

*(a)* $\mathrm{Ext}(G, \mathbb{Q}_p)$ *divisible and torsion-free* ([SA14a, Cor. 1.5]).

*(b)* $\mathrm{Ext}(G, X) = 0$ *for all divisible $\sigma$-compact $X \in \mathcal{L}$ if and only if* $G \cong \mathbb{R}^n \times G'$ *with $G'$ containing an open compact subgroup* $K \cong \prod_{\alpha<\alpha_0} \mathbb{Z}/p_\alpha^{r_\alpha}\mathbb{Z} \times \prod_{\beta<\beta_0} \mathbb{Z}_{p_\beta}^{\gamma_\beta}$ *where* $\alpha_0, \beta_0 \in \omega$, $\{\gamma_\beta : \beta < \beta_0\}$ *is a family of arbitrary ordinal numbers and* $p_\alpha, p_\beta \in \mathbb{P}$ ([SA14b, Th. 2.7]).

*(c) if $G$ is compact and torsion then* $\mathrm{Ext}(G, X) = 0$ *for every $X \in \mathcal{L}$ divisible and torsion-free* ([SA14a, Th. 1.6]).

Let us see how we can improve these results by making use of (3.2.4), (3.3.2) and (3.4.1).

*(a') Given a locally precompact abelian group $G$,* $\mathrm{Ext}(G, \prod_{p\in\mathbb{P}} \mathbb{Q}_p^{\alpha_p})$ *is a divisible, torsion-free group for every collection of ordinal numbers* $\{\alpha_p : p \in \mathbb{P}\}$.

*Proof.* According to (3.4.1)

$$\mathrm{Ext}(G, \textstyle\prod_{p\in\mathbb{P}} \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p\in\mathbb{P}} \mathrm{Ext}(G, \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p\in\mathbb{P}} \mathrm{Ext}(G, \mathbb{Q}_p)^{\alpha_p}.$$

Since $\mathbb{Q}_p$ is Čech-complete, by (3.2.4) $\mathrm{Ext}(G, \mathbb{Q}_p) \cong \mathrm{Ext}(\varrho G, \mathbb{Q}_p)$ for every $p \in \mathbb{P}$ and

$$\mathrm{Ext}(G, \textstyle\prod_{p\in\mathbb{P}} \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p\in\mathbb{P}} \mathrm{Ext}(\varrho G, \mathbb{Q}_p)^{\alpha_p}.$$

$\mathrm{Ext}(\varrho G, \mathbb{Q}_p)$ is divisible and torsion-free by (a), then $\mathrm{Ext}(G, \prod_{p\in\mathbb{P}} \mathbb{Q}_p^{\alpha_p})$ is also divisible and torsion-free. □

*(b') Let be $G$ a locally precompact abelian group and let $\mathcal{D}$ be the class of all groups of the form* $X = \prod_{\alpha<\kappa} X_\alpha$ *where $X_\alpha$ is a divisible $\sigma$-compact locally compact abelian group* $\forall \alpha < \kappa$. $\mathrm{Ext}(G, X) = 0$ *for all $X \in \mathcal{D}$ if and only if* $\varrho G = \mathbb{R}^n \times G'$ *where $n$ is a non-negative integer and $G'$ contains a compact open subgroup $K$ such that* $K \cong \prod_{\alpha<\alpha_0} \mathbb{Z}/p_\alpha^{r_\alpha}\mathbb{Z} \times \prod_{\beta<\beta_0} \mathbb{Z}_{p_\beta}^{\gamma_\beta}$ *where* $\alpha_0, \beta_0 \in \omega$, $\{\gamma_\beta : \beta < \beta_0\}$ *is a family of arbitrary ordinal numbers and* $p_\alpha, p_\beta \in \mathbb{P}$.

*Proof.* Suppose that a locally precompact abelian group $G$ has the property that $\mathrm{Ext}(G, X) = 0$ for all $X \in \mathcal{D}$. In particular for every divisible $\sigma$-compact $X \in \mathcal{L}$, we have that $\mathrm{Ext}(G, X) = 0$. Since every group in $\mathcal{L}$ is Čech-complete, by (3.2.4), $\mathrm{Ext}(\varrho G, X) = \mathrm{Ext}(G, X) = 0$, for all divisible $\sigma$-compact $X \in \mathcal{L}$. According to (b), $\varrho G$ has the desired structure.

Conversely if $\varrho G$ has the properties described in the statement, in virtue of (b), for every for all $X \in \mathcal{L}$ divisible $\sigma$-compact we have that $\mathrm{Ext}(\varrho G, X) =$

0. By (3.2.4), $\mathrm{Ext}(G,X)=0$ for every $X$ divisible $\sigma$-compact in $\mathcal{L}$. Finally, from (3.4.1) we conclude that the same is true for every $X\in\mathcal{D}$. □

*(c') Let $G$ be a torsion group in $\mathcal{L}$ and let $H$ be product of divisible torsion-free groups in $\mathcal{L}$. Then* $\mathrm{Ext}(G,H)=0$.

*Proof.* Suppose that $H=\prod_{\alpha<\kappa}H_\alpha$ with $H_\alpha$ divisible, torsion-free and $H_\alpha\in\mathcal{L}$ for every $\alpha<\kappa$. Since $G$ is locally compact abelian and torsion we know by [HR62, 24.18] that $G$ contains an open compact subgroup $K$. It is clear that $K$ will be a torsion group too. Applying (3.4.1) and (3.3.2)

$$\mathrm{Ext}(G,H)\cong\mathrm{Ext}(G,\textstyle\prod_{\alpha<\kappa}H_\alpha)\cong\prod_{\alpha<\kappa}\mathrm{Ext}(G,H_\alpha)\cong\prod_{\alpha<\kappa}\mathrm{Ext}(K,H_\alpha).$$

Invoking (c), we conclude that $\mathrm{Ext}(K,H_\alpha)=0$ for every $\alpha<\kappa$. □

**(3.4.3) Lemma.** *Let $\{E_\alpha:0\to H_\alpha\overset{\imath_\alpha}{\to}X_\alpha\overset{\pi_\alpha}{\to}G_\alpha\to 0:\alpha<\omega\}$ be a countable family of extensions of topological abelian groups. Consider the coproducts $\bigoplus_{\alpha<\omega}H_\alpha$, $\bigoplus_{\alpha<\omega}X_\alpha$, $\bigoplus_{\alpha<\omega}G_\alpha$ and the natural mappings $\bigoplus_{\alpha<\omega}\imath_\alpha:\bigoplus_{\alpha<\omega}H_\alpha\to\bigoplus_{\alpha<\omega}X_\alpha$ and $\bigoplus_{\alpha<\omega}\pi_\alpha:\bigoplus_{\alpha<\omega}X_\alpha\to\bigoplus_{\alpha<\omega}G_\alpha$. The sequence*

$$\bigoplus_{\alpha<\omega}E_\alpha:0\longrightarrow\bigoplus_{\alpha<\omega}H_\alpha\overset{\bigoplus_{\alpha<\omega}\imath_\alpha}{\longrightarrow}\bigoplus_{\alpha<\omega}X_\alpha\overset{\bigoplus_{\alpha<\omega}\pi_\alpha}{\longrightarrow}\bigoplus_{\alpha<\omega}G_\alpha\longrightarrow 0$$

*is an extension of topological abelian groups.*

*Proof.* The exactness of $\bigoplus_{\alpha<\omega}E_\alpha$ follows trivially from the exactness of each $E_\alpha$.

Let $\mathcal{I}_\beta:H_\beta\hookrightarrow\bigoplus_{\alpha<\omega}H_\alpha$ and $\mathcal{J}_\beta:X_\beta\hookrightarrow\bigoplus_{\alpha<\omega}X_\alpha$ be the canonical inclusions for every $\beta<\omega$. By definition of final group topology, since $(\bigoplus_{\alpha<\omega}\imath_\alpha)\circ\mathcal{I}_\beta$ and $(\bigoplus_{\alpha<\omega}\pi_\alpha)\circ\mathcal{J}_\beta$ are continuous for every $\beta<\omega$, we deduce that $\bigoplus_{\alpha<\omega}\imath_\alpha$ and $\bigoplus_{\alpha<\omega}\pi_\alpha$ are continuous.

The topology on each of the coproducts $\bigoplus_{\alpha<\omega}H_\alpha$, $\bigoplus_{\alpha<\omega}X_\alpha$ and $\bigoplus_{\alpha<\omega}G_\alpha$ coincides with the respective box topology (see (1.3.4.ii)). Thus, fix $(\prod_{\alpha<\omega}U_\alpha)\cap\bigoplus_{\alpha<\omega}H_\alpha\in\mathcal{N}_0(\bigoplus_{\alpha<\omega}H_\alpha)$ with $U_\alpha\in\mathcal{N}_0(H_\alpha)$ $\forall\alpha<\omega$. Since

$$\bigoplus_{\alpha<\omega}\imath_\alpha\Big(\Big(\prod_{\alpha<\omega}U_\alpha\Big)\cap\bigoplus_{\alpha<\omega}H_\alpha\Big)=\Big(\prod_{\alpha<\omega}\imath_\alpha(U_\alpha)\Big)\cap\bigoplus_{\alpha<\omega}X_\alpha$$

and $\imath_\alpha$ is relatively open for every $\alpha<\omega$, $\bigoplus_{\alpha<\omega}\imath_\alpha((\prod_{\alpha<\omega}U_\alpha)\cap\bigoplus_{\alpha<\omega}H_\alpha)$ is a neighborhood of 0 in $\bigoplus_{\alpha<\omega}\imath_\alpha(\bigoplus_{\alpha<\omega}H_\alpha)$. This proves that $\bigoplus_{\alpha<\omega}\imath_\alpha$ is relatively open. Analogously we prove that $\bigoplus_{\alpha<\omega}\pi_\alpha$ is open. ■

**(3.4.4) Theorem.** *Let $H$ be a topological abelian group and let $\bigoplus_{\alpha<\omega}G_\alpha$ a countable coproduct of topological abelian groups. Then* $\mathrm{Ext}(\bigoplus_{\alpha<\omega}G_\alpha,H)\cong\prod_{\alpha<\omega}\mathrm{Ext}(G_\alpha,H)$.

*Proof.* For each $\alpha < \omega$ consider the canonical inclusion $\mathcal{I}_\alpha : G_\alpha \to \bigoplus_{\beta<\omega} G_\beta$ and define

$$\begin{array}{rccc} \phi : & \mathrm{Ext}(\bigoplus_{\alpha<\omega} G_\alpha, H) & \longrightarrow & \prod_{\alpha<\omega} \mathrm{Ext}(G_\alpha, H) \\ & [E] & \longmapsto & \big([E\mathcal{I}_\alpha]\big)_{\alpha<\omega} \end{array}$$

According to (2.2.6), $\phi$ is a homomorphism of abelian groups. Let us see that $\phi$ is an isomorphism.

To see that $\phi$ is one-to-one pick an extension $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} \bigoplus_{\alpha<\omega} G_\alpha \to 0$ and suppose that $E\mathcal{I}_\beta$ splits for every $\beta < \omega$. Pick $\beta < \omega$. Take the sequence $E_\beta : 0 \to H \overset{\imath}{\to} \pi^{-1}(G_\beta) \overset{\pi_{|\pi^{-1}(G_\beta)}}{\to} G_\beta \to 0$. Since $\imath(H) \subset \pi^{-1}(G_\beta)$, $E_\beta$ is an exact sequence. Since $\pi_{|\pi^{-1}(G_\beta)}$ is open (see [Bou66, Prop 2, Chapter 5.1]) it follows that $E_\beta$ is an extension of topological abelian groups. For every $\beta < \omega$ the following diagram is commutative:

$$\begin{array}{lccccccccc} E : & 0 & \longrightarrow & H & \overset{\imath}{\longrightarrow} & X & \overset{\pi}{\longrightarrow} & \bigoplus_{\alpha<\omega} G_\alpha & \longrightarrow & 0 \\ & & & \Vert & & \uparrow & & \uparrow{\scriptstyle \mathcal{I}_\beta} & & \\ E_\beta : & 0 & \longrightarrow & H & \overset{\imath}{\longrightarrow} & \pi^{-1}(G_\beta) & \overset{\pi_{|\pi^{-1}(G_\beta)}}{\longrightarrow} & G_\beta & \longrightarrow & 0 \end{array}$$

Applying (2.2.6) we deduce that $E_\beta$ is equivalent to the pull-back extension $E\mathcal{I}_\beta$. Then $E_\beta$ splits. By (2.1.7) for every $\alpha < \omega$ there exist a continuous homomorphism $R_\alpha : G_\alpha \to \pi^{-1}(G_\alpha)$ with $\pi \circ R_\alpha = \mathrm{Id}_{G_\alpha}$. Consider the homomorphism $R : \bigoplus_{\alpha<\omega} G_\alpha \to X$ defined by $R((g_\alpha)_{\alpha<\omega}) = \sum_{\alpha<\omega} R_\alpha(g_\alpha)$. By de definition of the coproduct topology, $R$ is continuous. Since

$$\pi\Big(R\big((g_\alpha)_{\alpha<\omega}\big)\Big) = \pi\big(\sum_{\alpha<\omega} R_\alpha(g_\alpha)\big) = \sum_{\alpha<\omega} \pi\big(R_\alpha(g_\alpha)\big) = (g_\alpha)_{\alpha<\omega}$$

we obtain that $\pi \circ R = \mathrm{Id}_{\bigoplus_{\alpha<\omega} G_\alpha}$ and $E$ splits.

Let us check that $\phi$ is onto. Pick a family of extensions of topological abelian groups $\{\mathcal{E}_\alpha : 0 \to H \overset{\imath_\alpha}{\to} X_\alpha \overset{\pi_\alpha}{\to} G_\alpha \to 0 : \alpha < \omega\}$. From (3.4.3) we deduce that the sequence

$$\bigoplus_{\alpha<\omega} \mathcal{E}_\alpha : 0 \longrightarrow \bigoplus_{\alpha<\omega} H \overset{\bigoplus_{\alpha<\omega} \imath_\alpha}{\longrightarrow} \bigoplus_{\alpha<\omega} X_\alpha \overset{\bigoplus_{\alpha<\omega} \pi_\alpha}{\longrightarrow} \bigoplus_{\alpha<\omega} G_\alpha \longrightarrow 0$$

is an extension of topological abelian groups. Taking $P : \bigoplus_{\alpha<\omega} H \to H$; $P(h_\alpha)_{\alpha<\omega} \mapsto \sum_{\alpha<\omega} h_\alpha$ and the push-out extension $P \bigoplus_{\alpha<\omega} \mathcal{E}_\alpha$ we obtain the commutative diagram:

$$\begin{array}{lccccccccc} \mathcal{E}_\beta : & 0 & \longrightarrow & H & \overset{\imath_\beta}{\longrightarrow} & X_\beta & \overset{\pi_\beta}{\longrightarrow} & G_\beta & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{\alpha<\omega} \mathcal{E}_\alpha : & 0 & \longrightarrow & \bigoplus_{\alpha<\omega} H & \overset{\bigoplus \imath_\alpha}{\longrightarrow} & \bigoplus_{\alpha<\omega} X_\alpha & \overset{\bigoplus \pi_\alpha}{\longrightarrow} & \bigoplus_{\alpha<\omega} G_\alpha & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle P} & & \downarrow & & \Vert & & \\ P\bigoplus_{\alpha<\omega} \mathcal{E}_\alpha : & 0 & \longrightarrow & H & \longrightarrow & PO & \longrightarrow & \bigoplus_{\alpha<\omega} G_\alpha & \longrightarrow & 0 \end{array} \tag{E29}$$

From (2.2.6) and the commutativity of (E29) follows that $\mathcal{E}_\beta$ is equivalent to $(P \bigoplus_{\alpha<\omega} \mathcal{E}_\alpha)\mathcal{I}_\beta$ for every $\beta < \omega$ and therefore $\phi([P \bigoplus_{\alpha<\omega} \mathcal{E}_\alpha]) = ([\mathcal{E}_\beta])_{\beta<\omega}$. ■

**(3.4.5) Uncountable coproducts and (3.4.4).** It would be interesting to find out if (3.4.4) is true for uncountable coproducts of topological abelian groups. In (3.4.3) we used that for every countable family of topological abelian groups $\{G_\alpha : \alpha < \omega\}$, the coproduct topology on the direct sum $\bigoplus_{\alpha<\omega} G_\alpha$ coincides with the box topology. The author does not know any way to avoid the use of this fact and thus generalize (3.4.4).

It is worth mentioning that Fulp and Griffith proved in [FG71a, Th. 2.13] that if $\{G_\alpha : \alpha < \kappa\}$ is a family of groups in $\mathcal{L}$ such that $G_\alpha$ is discrete for all but a finite number of $\alpha < \kappa$, then

$$\operatorname{Ext}((\bigoplus_{\alpha<\kappa} G_\alpha, \tau_{\text{box}}), H) \cong \prod_{\alpha<\kappa} \operatorname{Ext}(G_\alpha, H)$$

where $H \in \mathcal{L}$ and $\tau_{\text{box}}$ is the box topology.

## §3.5 Ext and quotients

**(3.5.1) Theorem.** *Let $G$ and $H$ be topological abelian groups and let $M$ be a closed subgroup of $G$.*

*(i) Suppose that $\operatorname{Ext}(G/M, H) = 0$. Then every continuous homomorphism of $M$ to $H$ extends to a continuous homomorphism of $G$ to $H$.*

*(ii) Suppose that every continuous homomorphism of $M$ to $H$ extends to a continuous homomorphism of $G$ to $H$. Then $\operatorname{Ext}(G, H) = 0 \Rightarrow \operatorname{Ext}(G/M, H) = 0$.*

*Proof.* (i). Pick any continuous homomorphism $f : M \to H$ and consider the natural extension $E : 0 \to M \hookrightarrow G \to G/M \to 0$. Invoking the push-out extension $fE$ as in (2.2.2) we obtain the following commutative diagram:

$$\begin{array}{lccccccccc} E: & 0 & \longrightarrow & M & \hookrightarrow & G & \longrightarrow & G/M & \longrightarrow & 0 \\ & & & \downarrow f & & \downarrow s & & \| & & \\ fE: & 0 & \longrightarrow & H & \xrightarrow[r]{} & PO & \longrightarrow & G/M & \longrightarrow & 0 \end{array} \qquad \text{(E30)}$$

Since $\operatorname{Ext}(G/M, H) = 0$, $fE$ splits and by (2.1.7) there exists a continuous homomorphism $R : PO \to H$ such that $R \circ r = \operatorname{Id}_H$. In view of (E30) the homomorphism $R \circ s : G \to H$ is the desired extension of $f$.

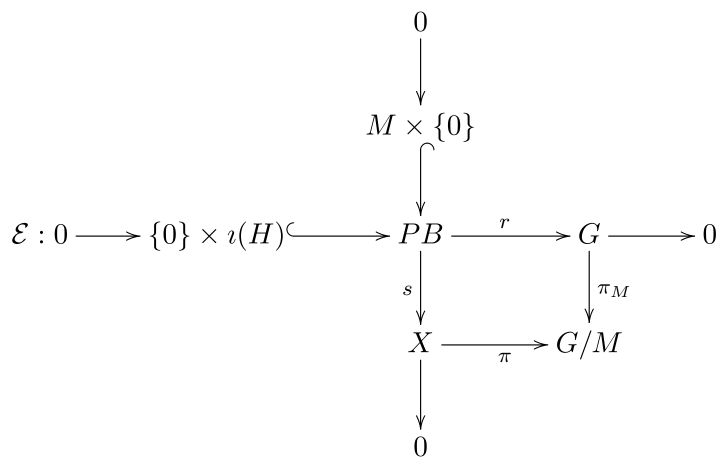
(ii). Fix an extension $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G/M \to 0$. Let $\pi_M : G \to G/M$ be the canonical projection and let

$$PB = \{(g, x) \in G \times X : \pi_M(g) = \pi(x)\} \tag{E31}$$

be the pull-back of $\pi_M$ and $\pi$ (see (2.2.5)). Defining $r : PB \to G;\ (g, x) \mapsto g$ and $s : PB \to X;\ (g, x) \mapsto x$,

$$\begin{aligned}
\ker r &= \{(g, x) \in PB : r(g, x) = 0\} = \{(g, x) \in PB : g = 0\} \\
&= \{(0, x) \in G \times X : 0 + M = \pi(x)\} \\
&= \{0\} \times \imath(H), \\
\ker s &= \{(g, x) \in PB : s(g, x) = 0\} = \{(g, x) \in PB : x = 0\} \\
&= \{(g, 0) \in G \times X : 0 + M = \pi_M(g)\} \\
&= M \times \{0\},
\end{aligned}$$

which gives us the following commutative diagram




Since $\pi_M$ and $\pi$ are onto, using (E31) one obtains that $r((U \times X) \cap PB) \supset U$ and $s((G \times V) \cap PB) \supset V$ for every $U \in \mathcal{N}_0(G)$ and $V \in \mathcal{N}_0(X)$; hence $r$ and $s$ are onto and open. Thus both short sequences are extensions of topological abelian groups.

Consider the continuous homomorphism $\varphi : \{0\} \times \imath(H) \to H$ defined by $\varphi(0, \imath(h)) = h$. Since $\text{Ext}(G, H) = 0$ and $H \cong \{0\} \times \imath(H)$, $\mathcal{E}$ splits. Therefore, we can find a continuous homomorphism $P : PB \to \{0\} \times \imath(H)$ satisfying $P(0, \imath(h)) = (0, \imath(h))$. Define $\psi = P \circ \varphi : PB \to H$, clearly $\psi(0, x) = \varphi(0, x)$ for every $x \in \imath(H)$. Consider, for every $m \in M$, $\widetilde{\psi}(m) = \psi(m, 0)$ (note that if $m \in M$ then $(m, 0) \in PB$).

By assumption there exists $\sigma : G \to H$ with $\sigma_{|M} = \widetilde{\psi}$ i.e. $\sigma(m) = \psi(m, 0)$ for every $m \in M$. Now define the continuous homomorphism $\gamma : PB \to H$ as follows: $\gamma(g, x) = \psi(g, x) - \sigma(g)$. Note that

$$\gamma(m, 0) = \psi(m, 0) - \sigma(m) = \psi(m, 0) - \psi(m, 0) = 0. \tag{E32}$$

In view of (E32) we can construct a well-defined continuous homomorphism $\mathcal{P} : X \to H$ as $\mathcal{P}(x) = \gamma(g, x)$. Finally, since $\mathcal{P}(\imath(h)) = \gamma(0, \imath(h)) = \psi(0, \imath(h)) - 0 = \varphi(0, \imath(h)) = h$ we conclude that $E$ splits.

■

**(3.5.2) Corollary.** *A topological abelian group $G$ satisfies that $\mathrm{Ext}(G, \mathbb{T}) = 0$ if and only if whenever $X$ is a topological abelian group and $H$ is a closed subgroup of $X$ with $X/H \cong G$, then $H$ is dually embedded.*

**(3.5.3) Lemma.** *Let $X$ be and $G$ be topological abelian groups. Suppose that $X$ is Hausdorff and that $\pi : X \to G$ is a continuous and open epimorphism.*

*(i) If $\ker \pi$ is compact then $\pi$ is perfect.*

*(ii) If $\ker \pi$ is locally compact then there exists $U \in \mathcal{N}_0(X)$ such that $\pi(\overline{U})$ is closed in $G$ and $\pi_{|\overline{U}} : \overline{U} \to \pi(\overline{U})$ is perfect.*

*Proof.* This is [AT08, Th. 1.5.7] and [AT08, Th. 3.2.2]. ■

**(3.5.4) Admissible subgroups.** A subgroup $N$ of a topological group $G$ is called *admissible* if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the neutral element 0 in $G$ such that $U_{n+1}+U_{n+1}+U_{n+1} \subset U_n$, for each $n \in \omega$, and $N = \bigcap_{n\in\omega} U_n$.

*(i) Every neighborhood of the neutral element of a topological group contains an admissible subgroup.*

*Proof.* Let $U$ be symmetric neighborhood of 0 and call $U_0 = U$. For every $n < \omega$, consider $U_{n+1}$ a symmetric neighborhood of 0 such that $U_{n+1} + U_{n+1} + U_{n+1} \subset U_n$. The family $\{U_n : n \in \omega\}$ defines an admissible subgroup $N = \bigcap_{n\in\omega} U_n \subset U_0 = U$. □

*(ii) Every admissible subgroup is closed.*

*Proof.* Suppose that $\{U_n : n \in \omega\}$ defines an admissible subgroup $N = \bigcap_{n\in\omega} U_n$. Pick $x \in G$ such that $(x + W) \cap N \neq \emptyset$ for every $W \in \mathcal{N}_0(G)$. In particular $(x + U_n) \cap N \neq \emptyset$ for every $n \in \omega$. Since $(x + U_n) \cap N = (x + U_n) \cap \bigcap_{n\in\omega} U_n \neq \emptyset$, $(x + U_n) \cap U_n \neq \emptyset$ and $x \in U_n + U_n \subset U_{n-1}$. We can repeat this argument for every $n$, then $x \in N$. □

*(iii) The intersection of two admissible subgroups is admissible.*

*Proof.* Let $\{U_n : n < \omega\}$ and $\{V_n : n < \omega\}$ be sequences of neighbourhoods of 0 in $G$ determining admissible subgroups $N_1$ and $N_2$ respectively. It is clear that

$$(U_{n+1} \cap V_{n+1}) + (U_{n+1} \cap V_{n+1}) + (U_{n+1} \cap V_{n+1}) \subset U_n \cap V_n.$$

Hence the sequence $\{U_n \cap V_n : n < \omega\}$ determines the admissible subgroup $\bigcap_{n\in\omega}(U_n \cap V_n) = \bigcap_{n\in\omega} U_n \cap \bigcap_{n\in\omega} V_n = N_1 \cap N_2$. □

*(iv) If $f : G \to H$ is a continuous homomorphism of topological abelian groups and $N$ is an admissible subgroup of $H$ then $f^{-1}(N)$ is an admissible subgroup of $G$.*

*Proof.* Suppose that $\{U_n : n < \omega\}$ determines the admissible subgroup $N \le H$. Since $U_{n+1} + U_{n+1} + U_{n+1} \subset U_n$ for every $n \in \omega$,

$$f^{-1}(U_{n+1})+f^{-1}(U_{n+1})+f^{-1}(U_{n+1}) = f^{-1}(U_{n+1}+U_{n+1}+U_{n+1}) \subset f^{-1}(U_n)$$

and it follows that the sequence $\{f^{-1}(U_n) : n < \omega\}$ defines the admissible subgroup

$$\bigcap_{n\in\omega} f^{-1}(U_n) = f^{-1}\Big(\bigcap_{n\in\omega} U_n\Big) = f^{-1}(N)$$

□

**(3.5.5) Lemma.** *Let $\pi\colon X \to G$ be continuous and open epimorphism of topological abelian groups, where $\ker\pi$ is a locally compact subgroup of $X$. There exists an admissible subgroup $N_0$ of $X$ such that for every closed subgroup $N$ of $X$ contained in $N_0$, the image $\pi(N)$ is closed in $G$ and there exists a continuous and open epimorphism $\varphi_N\colon X/N \to G/\pi(N)$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & G \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_{\pi(N)}} \\ X/N & \xrightarrow{\ \varphi_N\ } & G/\pi(N) \end{array}$$

*commutes and $\ker\varphi_N = \pi_N(\ker\pi)$, where $\pi_N\colon X \to X/N$ and $\pi_{\pi(N)}\colon G \to G/\pi(N)$ are the canonical projections.*

*Proof.* Since $\ker\pi$ is a locally compact subgroup of $X$, (3.5.3) implies that there exists a closed neighborhood $W$ of 0 in $X$ such that $\pi_{|W}$ is a perfect map. Use (3.5.4.i) to construct an admissible subgroup $N_0$ of $X$ with $N_0 \subseteq W$.

Let $N$ be a closed subgroup of $X$ contained in $N_0$. Since the map $\pi_{|W}$ is closed and $N \subset N_0 \subset W$, we see that the subgroup $\pi(N)$ is closed in $G$. Let $f = \pi_{\pi(N)} \circ \pi : X \to G/\pi(N)$,

$$\ker f = \{x : \pi_{\pi(N)}(\pi(x)) = 0 + \pi(N)\} = \{x : \pi(x) \in \pi(N)\} = N + \ker\pi.$$

Since $\ker\pi_N = N \subset N + \ker\pi = \ker f$, the homomorphism $\varphi_N\colon X/N \to G/\pi(N)$ defined by $\varphi_N(x + N) = f(x)$ is well-defined. Notice that $f$ is open as a composition of two open homomorphisms, so $\varphi_N$ is continuous and open. Finally, it is clear that $\varphi_N$ is onto and $\ker\varphi_N = \pi_N(N+\ker\pi) = \pi_N(\ker\pi)$. ■

**(3.5.6) Lemma.** *Let $\pi\colon X \to G$ be a continuous open homomorphism of topological abelian groups. If the kernel of $\pi$ is locally compact, then* *(i) There exists an admissible subgroup $N_0$ of $X$ such that $\pi(N)$ is an admissible subgroup of $G$, for each admissible subgroup $N$ of $X$ contained in $N_0$.*

*(ii) Let $\mathcal{F}$ be a cofinal subfamily of the family of admissible subgroups of $G$ ordered by inverse inclusion[1]. For every admissible subgroup $N$ of $X$, there exists an admissible $N' \subset N$ with $\pi(N') \in \mathcal{F}$.*

*Proof.* We can assume without loss of generality that $\pi(X) = G$, otherwise we replace $G$ with its open subgroup $\pi(X)$.

(i). Since $\ker \pi$ is a locally compact subgroup of $X$, it follows from (3.5.3) that there exists $U_0 \in \mathcal{N}_0(X)$ in $X$ such that the restriction of $\pi$ to $\overline{U_0}$ is a perfect map. Let $\{U_n : n \in \omega\}$ be a family of symmetric neighborhoods such that $U_{n+1} + U_{n+1} + U_{n+1} \subset U_n$ for each $n \in \omega$. Let us see $N_0 = \bigcap_{n\in\omega} U_n$ is the required admissible subgroup of $X$.

Indeed, let $N$ be an arbitrary admissible subgroup of $X$ contained in $N_0$. Take $\{V_n : n \in \omega\}$ a sequence of symmetric neighborhoods of 0 such that $V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$ for each $n \in \omega$ and $N = \bigcap_{n\in\omega} V_n$. The neighborhoods $O_n = V_n \cap U_n$, with $n \in \omega$, satisfy $O_{n+1} + O_{n+1} + O_{n+1} \subset O_n$ and $N = N_0 \cap N = \bigcap_{n\in\omega} O_n$. Then $W_n = \pi(O_n)$ is an open symmetric neighborhood of the neutral element in $G$ and $W_{n+1} + W_{n+1} + W_{n+1} \subset W_n$, for each $n \in \omega$. It is clear that $P = \bigcap_{n\in\omega} W_n$ is an admissible subgroup of $G$. To finish the proof of (i) it suffices to verify that $\pi(N) = P$.

It follows from the choice of the sets $O_n$ that $\overline{O_{n+1}} \subset O_n$, for each $n \in \omega$. In particular $N = \bigcap_{n\in\omega} \overline{O_n}$. Take any point $y \in P$. Then $\pi^{-1}(y) \cap O_n \neq \emptyset$ for each $n \in \omega$. As $O_0 \subset U_0$ and the map $\pi_{|\overline{U_0}}$ is perfect, the set $\overline{O_0} \cap \pi^{-1}(y)$ is compact. The family of closed sets $\{\pi^{-1}(y) \cap \overline{O_n} : n < \omega\}$ has the finite intersection property, therefore $\emptyset \neq \pi^{-1}(y) \cap \bigcap_{n\in\omega} \overline{O_n} = \pi^{-1}(y) \cap N$. This proves the equality $\pi(N) = P$.

(ii). Fix an admissible subgroup $N \leq X$. Take an admissible subgroup $N_0$ of $X$ as in (i). Then $\pi(N_0 \cap N)$ is an admissible subgroup of $G$. Since $\mathcal{F}$ is cofinal, there exists $P \in \mathcal{F}$ such that $P \subset \pi(N_0 \cap N)$. Put $N' = N_0 \cap N \cap \pi^{-1}(P)$. Then $N' \subset N$ is an admissible subgroup of $X$ and $\pi(N') = \pi(N_0 \cap N) \cap P = P$. ■

**(3.5.7) Theorem.** *Let $M$ be a metrizable, locally compact abelian group. Let $G$ be a topological abelian group and $\mathcal{F}$ a cofinal subfamily of the family of admissible subgroups of $G$, ordered by inverse inclusion. If* $\mathrm{Ext}(G/P, M) = 0$ *for each $P \in \mathcal{F}$, then* $\mathrm{Ext}(G, M) = 0$.

[1] i.e. $\mathcal{F}$ satisfies that for every admissible $N \leq G$ there exists $N' \in \mathcal{F}$ such that $N' \subset N$.

*Proof.* Suppose that $E\colon 0 \to M \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ is a extension of topological abelian groups. Since $\ker \pi = \imath(M)$ is locally compact, we can find an admissible subgroup $N_2$ of $X$ as in (3.5.5). We will start proving the following fact:

($*$) *There exists an admissible subgroup $N_1$ of $X$ such that $N_1 \cap \imath(M) = \{0\}$ and $N_1 \subset N_2$.*

Indeed, since $\imath(M)$ is metrizable, we can construct a decreasing family $\{O_n : n < \omega\}$ of open symmetric neighborhoods of 0 in $X$ such that $\bigcap_{n<\omega}(O_n \cap \imath(M)) = \{0\}$. Take for every $n < \omega$, $V_n \in \mathcal{N}_0(X)$ open, symmetric and such that $V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$. Define $W_n = V_n \cap O_n$. Considering the admissible subgroup $N_1' = \bigcap_{n<\omega} W_n$ one easily sees that $N_1 = N_1' \cap N_2$ is an admissible subgroup (3.5.4.iii) which satisfies the desired properties. □

By (3.5.6.ii), we can find an admissible subgroup $N_0$ of $X$ such that $N_0 \subset N_1$ and $\pi(N_0) \in \mathcal{F}$. Let us take the sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of 0 in $X$ that witnesses the fact that $N_0$ is an admissible subgroup of $X$. Clearly $\overline{U_{n+1}} \subset U_n$ for each $n \in \omega$. Since the group $\imath(M)$ is locally compact and $N_0 \cap \imath(M) \subset N_1 \cap \imath(M) = \{0\}$, the family $\{U_n \cap \imath(M) : n < \omega\}$ forms a local base at 0 in $\imath(M)$ (see [Eng89, 3.1.5]).

Let $p\colon X \to X/N_0$ and $f\colon G \to G/\pi(N_0)$ be the canonical projections. As $N_0 \subset N_1 \subset N_2$, by (3.5.5) there exists a continuous open epimomorphism $\varphi : X/N_0 \to G/\pi(N_0)$ such that $f \circ \pi = \varphi \circ p$ and $\ker \varphi = p(\imath(M))$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & M & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} & & \\
E': 0 & \longrightarrow & p(\imath(M)) & \xrightarrow{\mathcal{I}} & X/N_0 & \xrightarrow{\varphi} & G/\pi(N_0) & \longrightarrow & 0
\end{array}$$

where $\mathcal{I}$ is the canonical inclusion and $\mathcal{P}(x) = p(\imath(x))$ for every $x \in M$. The sequence $E'$ is also an extension of topological abelian groups. Notice also that since $N_1 \cap \imath(M) = \{0\}$ (see ($*$)), $\mathcal{P}$ is one-to-one. Pick $U \in \mathcal{N}_0(M)$, since $\{\imath^{-1}(U_n) : n < \omega\}$ is a local base at 0 of $M$, there exists $n < \omega$ such that $\imath^{-1}(U_n) \subset U$. Since $N_0 + U_{n+1} \subset U_n \; \forall n < \omega$,

$$\mathcal{P}^{-1}\Big(p\big(\imath(M)\big) \cap p(U_{n+1})\Big) = i^{-1} \circ p^{-1}\Big(p\big(\imath(M)\big) \cap p(U_{n+1})\Big) \subset \imath^{-1}(U_n)$$

then

$$p\big(\imath(M)\big) \cap p(U_{n+1}) \subset \mathcal{P}(\imath^{-1}(U_n)) \subset \mathcal{P}(U).$$

Therefore $\mathcal{P}$ is open, hence a topological isomorphism.

Since $\pi(N_0) \in \mathcal{F}$, by hypothesis the extension $E'$ splits. (2.1.7) implies that there exists a continuous homomorphism $R\colon X/N_0 \to p(\imath(M))$ such that $R \circ \mathcal{I} = \mathrm{Id}_{p(\imath(M))}$. It is clear that the continuous homomorphism $S\colon X \to M$ defined by $S = \mathcal{P}^{-1} \circ R \circ p$ satisfies $S \circ \imath = \mathrm{Id}_M$. Hence the extension $E$ splits. ■

**(3.5.8) Notes.** (3.2.2) and (3.2.3) are [BCDT16, Prop. 3.9] and [BCDT16, Prop. 3.10] respectively. The results (3.2.4),(3.2.6), (3.3.2), (3.4.1), (3.4.2), (3.4.4) can be found in [Bel]. (3.4.4) is an improvement of [BCD13, Th. 24]. (3.5.1) was proven for $M = \mathbb{T}$ in [BCD13, Th. 21], furthermore, an analogous version of this result for completely metrizable topological vector spaces appears in [KPR84, Th. 5.2] and [Dom85, Lemma 4.1]. (3.5.2) is [BCD13, Cor. 23], (3.5.5) is [BCDT16, Lemma 3.3] and (3.5.6) is [BCDT16, Lemma 3.4]. (3.5.7) was proven in [BCDT16, Th. 3.5].

# Chapter 4

# The Ext group in the category of topological vector spaces

In this chapter we will focus on the extensions of topological vector spaces. We will use the techniques of the previous chapter to study the analogous object to Ext group in this class, which will be a vector space that we will denote by $\mathrm{Ext}_{\mathrm{TVS}}$. Finally, in §4.2 we will apply our knowledge of Ext and $\mathrm{Ext}_{\mathrm{TVS}}$ to Problem 4, continuing the work of Cattaneo ([Cat80]) and Cabello ([Cab04]).

## §4.1 The $\mathrm{Ext}_{\mathrm{TVS}}$ group

**(4.1.1) Extensions of topological vector spaces.** If $X, Y$ and $Z$ are topological vector spaces, a sequence $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ is called *an extension of topological vector spaces*[1] if it is an extension of topological abelian groups and the maps $\imath$ and $\pi$ are also linear.

Two extensions of topological vector spaces $E : 0 \to Y \to X \to Z \to 0$ and $E' : 0 \to Y \to X' \to Z \to 0$ are said to be *equivalent* if there exists a continuous linear mapping $T : X \to X'$ making commutative a diagram analogous to (E1).

*(i) Every continuous homomorphism $T : X \to X'$ between topological vector spaces is linear.*

*Proof.* Pick $x \in X$ and $\lambda \in \mathbb{R}$. Take $\{a_n/b_n\} \in \mathbb{Q}^\omega$ such that $a_n, b_n \in \mathbb{Z}$ and $\lim_{n\in\omega} a_n/b_n = \lambda$. Since $\lim_{n\in\omega}(a_n/b_n)x = \lambda x$, it suffices to show that for all $n < \omega$

$$T\left(\frac{a_n}{b_n}x\right) = \frac{a_n}{b_n}T(x). \tag{E33}$$

[1] Some authors prefer the notation *twisted sum of topological vector spaces*, especially in the context of $F$-spaces.

Since $T$ is a homomorphism

$$b_n T\Big(\frac{a_n}{b_n}x\Big) = T\Big(b_n \frac{a_n}{b_n}x\Big) = T(a_n x) = a_n T(x). \tag{E34}$$

Using the scalar multiplication in $X'$ and (E34) we obtain (E33). □

*(ii) Two extensions of topological vector spaces are equivalent if and only if they are equivalent when regarded as extensions of topological abelian groups (in the sense of (2.1.1)).*

*Proof.* This is a consequence of (i). □

*(iii) Let $G$ and $H$ be topological abelian groups. If $H$ is a topological vector space then* $\mathrm{Ext}(G,H)$ *has a natural structure of vector space.*

*Proof.* Since $\mathrm{Ext}(G,H)$ is an abelian group we only need to construct a multiplication by scalars. Suppose that $H$ is a topological vector space. Consider for every $r \in \mathbb{R}$ the topological isomorphism

$$\begin{array}{rccc} \phi_r : & H & \longrightarrow & H \\ & h & \longmapsto & rh \end{array}$$

and define

$$\begin{array}{rccc} \varphi : & \mathbb{R} \times \mathrm{Ext}(G,H) & \longrightarrow & \mathrm{Ext}(G,H) \\ & (r,[E]) & \longmapsto & [\phi_r E] \end{array}$$

By (E21)

$$\begin{aligned} \varphi(r,[E]+[E']) &= [\phi_r(E+E')] = [\phi_r(E)+\phi_r(E')] \\ &= \varphi(r,[E]) + \varphi(r,[E']) \end{aligned}$$

$$\begin{aligned} \varphi(r+r',[E]) &= [\phi_{r+r'}E] = [(\phi_r+\phi_{r'})E] = [\phi_r E + \phi_{r'} E] \\ &= \varphi(r,[E]) + \varphi(r',[E]). \end{aligned}$$

Using the uniqueness of the push-out sequence (1.1.4)

$$\varphi(rr',[E]) = [\phi_{rr'}E] = [\phi_r \circ \phi_{r'}E] = [\phi_r(\phi_{r'}E)] = \varphi(r,\varphi(r',[E])).$$

Since $\phi_1 = \mathrm{Id}_H$, $\varphi(1,[E]) = [\mathrm{Id}_H E] = [E]$. Consequently, $\varphi$ is a product by scalars on the abelian group $\mathrm{Ext}(G,H)$. □

We will define $\mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$ as the set of equivalence classes of extensions of topological vector spaces of the form $0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$. Considering the Baer sum as in (3.1.1), we can endow $\mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$ with the structure of abelian group.

*(iv) For every topological vector spaces $Y$ and $Z$,* $\mathrm{Ext}_{TVS}(Z,Y)$ *has a natural structure of vector space.*

*Proof.* Just notice that in this setting the map $\phi_r : Y \to Y;\ y \mapsto ry$ is a continuous linear mapping. It is easy to see that the push-out of an extension of topological vector spaces by a continuous linear mapping gives us an extension of topological vector spaces. Thus given $[E] \in \mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$, it is clear that $[\phi_r E] \in \mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$. Therefore, we can use the scalar multiplication $\varphi$ defined in (iii) to endow $\mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$ with a vector space structure. □

Note that the topological vector structure on $Z$ gives $\mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$ another vector space structure via the corresponding pull-backs (instead of push-outs). We will not consider this structure here.

**(4.1.2) Theorem.** *Let $Y, Z$ be topological vector spaces. Suppose that $Y$ is complete and metrizable; then*

*(i)* $\mathrm{Ext}_{TVS}(Z,Y) \cong \mathrm{Ext}_{TVS}(\varrho Z,Y)$.

*(ii) if $D$ is a dense subspace of $Z$ then* $\mathrm{Ext}_{TVS}(Z,Y) \cong \mathrm{Ext}_{TVS}(D,Y)$

*Proof.* (i) Since $Y$ is complete and metrizable, it is Čech-complete and we are in the conditions of (3.2.3). Notice that if we use (3.2.3) to complete an extension of topological vector spaces we obtain an extension of topological vector spaces. Having this in mind we can repeat the proof of (3.2.4) in this context.

(ii) Proceed as in (3.2.5) using (i) instead of (3.2.4). ■

**(4.1.3) Theorem.** *Let $Z$ be a topological vector space and let $\{Y_\alpha : \alpha < \kappa\}$ be a family of topological vector spaces. Then* $\mathrm{Ext}_{TVS}(Z, \prod_{\alpha<\kappa} Y_\alpha) \cong \prod_{\alpha<\kappa} \mathrm{Ext}_{TVS}(Z, Y_\alpha)$.

*Proof.* All the steps in the proof of (3.4.1) are applicable in the category of topological vector spaces so we can proceed in the same way. ■

## §4.2 “ Being a topological vector space” as a three space property

We will turn our attention to applying the techniques developed in the previous chapter to Problem 4 of Chapter 1, which we recall here:

**Problem 4.** *If $Y, Z$ are topological vector spaces and $E : 0 \to Y \to X \to Z \to 0$ is an extension of topological abelian groups, what properties of $Y$ and $Z$ do we need in order that $X$ admits a compatible topological vector space structure that transforms $E$ into an extension of topological vector spaces?*

**(4.2.1) Compatible topological vector space structures.** Let $X$ be a topological abelian group. Suppose that $\star : \mathbb{R} \times X \to X$ is a multiplication by scalars in $X$ such that $(X, \star)$ is a topological vector space that has the same underlying topological abelian group structure of $X$ (i.e. such that it has the same addition and the same topology). Then $(X, \star)$ is said to be a *compatible topological vector space structure* for $X$.

*Fact 1. Let $Y, Z$ be topological vector spaces and let $X$ be a topological abelian group. Suppose that $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ is an extension of topological abelian groups. The following are equivalent:*

*(i) There exists a compatible topological vector space structure $(X, \star)$ for $X$ making $\imath : Y \to (X, \star)$ and $\pi : (X, \star) \to Z$ linear maps (i.e. one that makes $E$ an extension of topological vector spaces).*

*(ii) $E$ is equivalent to an extension of topological vector spaces $\mathcal{E} : 0 \to Y \xrightarrow{\mathcal{I}} \mathcal{X} \xrightarrow{\mathcal{P}} Z \to 0$.*

*Proof.* $(i) \Rightarrow (ii)$. Consider the sequence $\mathcal{E} : 0 \to Y \xrightarrow{\imath} (X, \star) \xrightarrow{\pi} Z \to 0$. Since by assumption $E$ is an extension of topological abelian groups and $\imath : Y \to (X, \star)$ and $\pi : (X, \star) \to Z$ are linear mappings, it follows that $\mathcal{E}$ is an extension of topological vector spaces. The identity $X \to (X, \star)$; $x \mapsto x$ witnesses the equivalence of $E$ and $\mathcal{E}$.

$(ii) \Rightarrow (i)$. Suppose that $T : X \to \mathcal{X}$ is a topological isomorphism that makes the diagram

$$\begin{array}{lccccccccc} \mathcal{E}: & 0 & \longrightarrow & Y & \xrightarrow{\mathcal{I}} & \mathcal{X} & \xrightarrow{\mathcal{P}} & Z & \longrightarrow & 0 \\ & & & \| & & \uparrow T & & \| & & \\ E: & 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array} \tag{E35}$$

commutative. Define

$$\begin{array}{rccc} \star : & \mathbb{R} \times X & \longrightarrow & X \\ & (r, x) & \longmapsto & T^{-1}\big(r \cdot T(x)\big) \end{array}$$

Using that $T$ is an isomorphism one easily proves that $\star$ is a multiplication by scalars. By the commutativity of (E35),

$$r \star \imath(x) = T^{-1}\Big(rT\big(\imath(x)\big)\Big) = T^{-1}\big(r\mathcal{I}(x)\big) = T^{-1}\big(\mathcal{I}(rx)\big) = \imath(rx),$$

$$\pi(r \star x) = \pi\Big(T^{-1}\big(rT(x)\big)\Big) = \mathcal{P}\big(rT(x)\big) = r\mathcal{P}\big(T(x)\big) = r\pi(x),$$

hence under multiplication by scalars $\star$, $\imath$ and $\pi$ are linear. The continuity of $\star$ follows from the continuity of $T$ and $T^{-1}$. □

The following Fact, due to Cattaneo and Cabello, connects with Problem 4.

*Fact 2. Let $Y, Z$ be topological vector spaces and let $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ be an extension of topological abelian groups. $X$ admits a compatible topological vector space structure, making $E$ an extension of topological vector spaces, if any of the following conditions is satisfied:*

*(a) $Y$ is a Fréchet topological vector space and $Z$ is any metrizable complete topological vector space* [Cat80, Prop. 2].

*(b) $Y$ and $Z$ are a complete locally bounded topological vector spaces* [Cab04, Th. 4].

In (4.2.2) and (4.2.3) we will show that the completeness of $Z$ in (a) and (b) can be dropped.

**(4.2.2) Theorem.** *Let $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ be an extension of topological abelian groups. Suppose that $Y$ is a Fréchet topological vector space and $Z$ is a metrizable topological vector space. Then $X$ admits a compatible topological vector space structure in such a way that $E$ becomes an extension of (metrizable) topological vector spaces.*

*Proof.* Metrizability is a three space property (see [HR62, 5.38(e)]) hence $X$ is metrizable.

Since $Y$ is metrizable and complete, it is Čech-complete (see (1.3.12)) and we can apply (3.2.3) to deduce that the completed sequence $\varrho E : 0 \to Y \xrightarrow{\varrho\imath} \varrho X \xrightarrow{\varrho\pi} \varrho Z \to 0$ is an extension of topological abelian groups. According to (4.2.1.a) there exists a compatible topological vector space structure in $\varrho X$ and we can regard $\varrho E$ as an extension of topological vector spaces. Consider the canonical inclusion $\mathcal{I} : Z \to \varrho Z$. The following diagram is commutative:

$$\begin{array}{llllllllll} \varrho E : & 0 & \longrightarrow & Y & \xrightarrow{\varrho\imath} & \varrho X & \xrightarrow{\varrho\pi} & \varrho Z & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow{\scriptstyle \mathcal{I}} & & \\ E : & 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array}$$

Notice that since $\varrho E$ is now an extension of topological vector spaces and $\mathcal{I}$ is a linear mapping, when we construct the pull-back sequence $(\varrho E)\mathcal{I}$ we obtain an extension of topological vector spaces. In virtue of (2.2.7), $E$ is equivalent to $(\varrho E)\mathcal{I} : 0 \to Y \to PB \to Z \to 0$. Applying Fact 1 of (4.2.1) we complete the proof.

■

**(4.2.3) Theorem.** *Let $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ be an extension of topological abelian groups. Suppose that $Y$ is a complete locally bounded topological vector space and $Z$ is a locally bounded topological vector space. Then $X$ admits a topological vector space structure such that $E$ becomes an extension of (locally bounded) topological vector spaces.*

*Proof.* $X$ is locally bounded because local boundedness is a three space property (see [RD81a, Th. 3.2]). $Y$ is metrizable because every locally bounded Hausdorff topological vector space is metrizable. Hence $Y$ is Čech-complete and by (1.3.12) we can apply (3.2.3) to deduce that the completion $\varrho E : 0 \to Y \overset{\varrho\iota}{\to} \varrho X \overset{\varrho\pi}{\to} \varrho Z \to 0$ is an extension of topological abelian groups. Since $Z$ is locally bounded, its completion $\varrho Z$ is also locally bounded and we are in the conditions of (4.2.1.b). So $\varrho E$ can be regarded as an extension of topological vector spaces. From here proceed as in the proof of (4.2.2). ■

**(4.2.4) Corollary.** *Let $Y$ and $Z$ be topological vector spaces.*

*(i) If $Y$ is Fréchet and $Z$ is metrizable then* $\mathrm{Ext}(Z, Y) \cong \mathrm{Ext}_{TVS}(Z, Y)$.

*(ii) If $Y$ is complete locally bounded and $Z$ is locally bounded then* $\mathrm{Ext}(Z, Y) \cong \mathrm{Ext}_{TVS}(Z, Y)$

**(4.2.5) Corollary.** *Let $X$ be an abelian topological group and let $\pi : X \to Z$ be an open continuous homomorphism of $X$ onto a topological vector space $Z$. Then $X$ admits a compatible topological vector space structure such that $\pi$ becomes a linear mapping in any of the following situations:*

*(i)* $\ker \pi$ *is a Fréchet space and $Z$ is metrizable.*

*(ii)* $\ker \pi$ *is a complete locally bounded topological vector space and $Z$ is locally bounded.*

**(4.2.6) $\mathcal{K}$-spaces.** Kalton, Peck and Roberts studied in [KPR84] the extensions of topological vector spaces of the form $0 \to \mathbb{R} \to X \to Z \to 0$ where $Z$ is complete and metrizable (F-space). They introduced and studied the $\mathcal{K}$*-spaces* which are complete metrizable topological vector spaces $Z$ satisfying $\mathrm{Ext}_{\mathrm{TVS}}(Z, \mathbb{R}) = 0$. From (4.2.4.i) follows that:

- *A metrizable vector space $Z$ satisfies that* $\mathrm{Ext}(Z, \mathbb{R}) = 0$ *if and only if $\varrho Z$ is a $\mathcal{K}$-space.*

In this line of thinking, Domański investigated in [Dom85] a more general form of this problem. For a given topological vector space $Y$, he studied what he called the class $\mathcal{S}(Y)$ of topological vector spaces $Z$ such that $\mathrm{Ext}_{\mathrm{TVS}}(Z, Y) = 0$.

In view of (4.2.4.i) and (4.2.4.ii) and using Domański's terminology we obtain:

- *Let $Y$ be a Fréchet space and $Z$ be any metrizable topological vector space. Then $Z \in \mathcal{S}(Y)$ if and only if* $\mathrm{Ext}(Z, Y) = 0$.

- *Let $Y$ and $Z$ be locally bounded topological vector spaces. If $Y$ is complete then $Z \in \mathcal{S}(Y)$ if and only if* $\mathrm{Ext}(Z, Y) = 0$.

**(4.2.7) Notes.** The results (4.1.2), (4.1.3), (4.2.2) and (4.2.3) can be found in [Bel].

# Chapter 5

# Cross-sections

The notion of cross-section has been considered in many different contexts across topological and abstract algebra. Most authors define cross-sections simply as right inverses of epimorphisms. The study of cross-sections in topological groups started with the work of Comfort, Hernández and Trigos-Arrieta [CHTA01], where they use the notion of continuous cross-section as a way to approach several problems related with the Bohr topology on an abelian group (see [CHTA01, def. 4]).

This chapter is devoted to the study of cross-sections the context of topological abelian groups. We will be interested in those cross-sections that are continuous at the identity or globally continuous.

We will start §5.1 introducing this notion and its basic properties, specially in regard to extensions of topological abelian groups. We will use Michael's selection theorem to find conditions under which an extension admits a continuous cross-section. In §5.2 we will study situations in which an extension of vector spaces admits a continuous cross-section.

## §5.1 Topological abelian groups and cross-sections

**(5.1.1) Definition.** Let $\pi : X \to G$ be a continuous and open epimorphism of topological abelian groups. A map $s : G \to X$ is called *a cross-section of* $\pi$ if it satisfies that $\pi \circ s = \mathrm{Id}_G$.

**(5.1.2) Definition.** A subgroup $H$ of a topological abelian group $G$ is called a *ccs-subgroup* if the natural projection $G \to G/H$ admits a continuous cross-section.

**(5.1.3) Examples.** (i) Let $G^\#$ denote an abelian group $G$ endowed with its Bohr topology i.e. the topology induced by $\mathrm{Hom}(G, \mathbb{T})$. $\mathbb{Z}^\#$ is a ccs-subgroup of $\mathbb{Q}^\#$ (see [CHTA01, Th. 24]). Dikranjan proved in [Dik02, Example 3.9] that $\mathbb{Z}_p^{\#}$ is a ccs-subgroup of every topological abelian group of the form $G^{\#}$ containing $\mathbb{Z}_p^{\#}$.

(ii) Let $H$ be an open subgroup of a topological group $G$, then $H$ is a ccs-subgroup of $G$. Indeed consider the natural projection $\pi : G \to G/H$ and take any map $s : G/H \to G$ satisfying $s(g + H) = \pi^{-1}(g)\ \forall g \in G$. Since $H$ is open, $G/H$ is discrete and consequently $s$ is continuous, thus it is a continuous cross-section. In particular $\mathbb{Z}_p$ is a ccs-subgroup of $\mathbb{Q}_p$.

(iii) Let $G$ be a locally compact separable metric abelian group and $H$ any closed subgroup of $G$. Suppose that $G$ is has finite dimension ([Str06, §33]). Then the natural projection $\pi : G \to G/H$ admits a cross-section continuous on a neighborhood of the neutral element. This result was proven by Mostert in [Mos53, Th. 3].

**(5.1.4) Extensions and cross-sections.** Given an extension of topological abelian groups $E : 0 \to H \to X \stackrel{\pi}{\to} G \to 0$, we will say that $E$ admits a cross-section continuous at 0 (resp. continuous) if so does $\pi$. It is clear that if $E$ admits a cross-section $s : G \to X$ (either continuous or continuous at 0) we can always suppose without loss of generality that $s(0) = 0$.

The aim of this chapter can be expressed as follows:

*Problem. Find conditions on topological abelian groups $H$ and $G$ which guarantee that every extension of the form $0 \to H \to X \to G \to 0$ admits a continuous cross-section (or at least continuous at $0$).*

Results (5.1.9), (5.1.16) and (5.2.2) are partial answers to the previous problem.

**(5.1.5) Lemma.** *Let $E : 0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ be an extension of topological groups. The following are equivalent*

*(i) There exists a map $r : X \to H$ continuous at the origin (respectively continuous) satisfying $r(\imath(h)) = h$ and $r(x + \imath(h)) = r(x) + h$ for every $h \in H, x \in X$.*

*(ii) $E$ admits a cross-section continuous at $0$ (continuous).*

*(iii) There exists a bijection $\phi : H \times G \to X$ continuous at the origin with inverse continuous at the origin (homeomorphism) and such that $\phi(h, 0) = \imath(h)$, $\pi(\phi(h, g)) = g$ for all $h \in H, g \in G$.*

*Proof.* We will present the proof for the case in which $E$ admits a cross-section continuous at 0, if it admits a globally continuous one the argument is analogous.

(i)⇒(ii). Consider the map $s : G \to X;\ \pi(x) \mapsto x - \imath(r(x))$. $s$ is well-defined because taking $\pi(x) = \pi(x + \imath(h))$, we obtain:

$$s(\pi(x + \imath(h))) = x + \imath(h) - \imath(r(x + \imath(h))) = x - \imath(r(x)) = s(\pi(x)).$$

To check that $s$ is continuous at 0, notice that since $G$ has the quotient topology induced by $\pi$, it suffices to show that $s \circ \pi$ is continuous at 0, which is trivial. From the definition of $s$ follows that it is a cross-section.

(ii)⇒(iii). Since $s$ is continuous at 0, the map $\phi : H \times G \to X;\ (h, g) \mapsto s(g) + \imath(h)$ is continuous at 0. Notice that for all $g \in G, h \in H$, we have that $\phi(h, 0) = s(0) + \imath(h) = \imath(h)$ and $\pi(\phi(h, g)) = \pi(s(g)) + \pi(\imath(h)) = g + 0 = g$. For every $x \in X$, since $\pi(x - s(\pi(x))) = 0$, we deduce that $x - s(\pi(x)) \in \imath(H)$. The inverse of $\phi$ is the map $x \mapsto (\imath^{-1}(x - s(\pi(x))), \pi(x))$, which is well-defined and continuous at 0.

(iii)⇒(i). Define $r : X \to H$ as $r(x) = \imath^{-1}(x - \phi(0, \pi(x)))$. $r$ is continuous at 0 because $\imath$ is an embedding and $\phi$ is continuous at 0. Since $r(\imath(h)) = \imath^{-1}(\imath(h) - \phi(0, 0)) = h$ and

$$\begin{aligned} r(x + \imath(h)) &= \imath^{-1}\Big(x + \imath(h) - \phi\big(0, \pi(x + \imath(h))\big)\Big) \\ &= \imath^{-1}\Big(x - \phi(0, \pi(x)) + \imath(h)\Big) = r(x) + h, \end{aligned}$$

we conclude that $r$ is the desired map. ■

**(5.1.6) Proposition.** *Let $\pi : X \to G$ be a continuous open epimorphism of topological abelian groups. If $X$ is metrizable then $\pi$ admits a cross-section continuous at 0. In particular every extension of the form $0 \to H \to X \to G \to 0$ where $H$ and $G$ are metrizable admits cross-section continuous at 0.*

*Proof.* Note that in order to define $s$ with $\pi \circ s = \mathrm{id}_G$, we simply must choose for every $g \in G$ an element $x \in \pi^{-1}(g)$, which is a nonempty set since $\pi$ is onto. Let us see that it can be done in such a way that the map thus obtained is continuous at zero.

Let $\{U_n : n < \omega\}$ be a decreasing basic sequence of neighborhoods of zero in $X$, where $U_1 = X$. Due to the continuity of $\pi$, we have $\bigcap_{n \in \mathbb{N}} \pi(U_n) = \{0\}$.

Let $s$ take the value 0 on $g = 0$. For any $g \neq 0$, by the previous paragraph we can choose $n$ and $x$ with $\pi(x) = g,\ x \in U_n,\ g \notin \pi(U_{n+1})$, and define $s(g) = x$. Now fix $m < \omega$, we must find $V \in \mathcal{N}_0(G)$ with $s(V) \subset U_m$. Since $\pi$ is open there exists $V \in \mathcal{N}_0(G)$ with $\pi(U_m) \supseteq V$. Fix $g \in V$ and let us show that $s(g) \in U_m$. If $s(g) = 0$ this is trivial. Otherwise $s(g) = x$ with $\pi(x) = g,\ x \in U_n,\ g \notin \pi(U_{n+1})$ for some $n$. Then $g \in V \subset \pi(U_m)$, hence $m \leq n$ and $x \in U_n \subset U_m$. ■

**(5.1.7) Lower semicontinuous maps.** Let $Z$, $M$ be topological spaces. $\exp(M)$ stands for the set of all closed non-empty subsets of $M$. A mapping $q : Z \to \exp(M)$ is called *lower semicontinuous* if for every $V$ open subset of $M$, the set

$$V_q = \{z \in Z : q(z) \cap V \neq \emptyset\}$$

is open in $Z$. A map $S : Z \to M$ such that $S(z) \in q(z)\ \forall z \in Z$ is called a *selection* of $q$.

The next fact will be useful in the following constructions:

*($*$) Suppose that $f : M \to Z$ is a continuous onto and open map and that $Z$ is Hausdorff. Then the map $q : Z \to \exp(M)$ defined by $q(z) = f^{-1}(z)$ is lower semicontinuous.*

*Proof.* Pick $W$ open in $M$, let us show that $W_q = \{z \in Z : f^{-1}(z) \cap W \neq \emptyset\}$ is open in $Z$. Choose $z \in W_q$, since $f$ is open, $f(W)$ is open in $Z$. Since $f^{-1}(z) \cap W \neq \emptyset$, $z \in f(W)$ and $f(W)$ is a neighborhood of $z$. Since $f(W) \subset W_q$, $W_q$ is open in $Z$. □

We proceed to invoke the version of Michael's Selection Theorem that we will need (for more detail see [Mic56, Th. 1.2] and [Mic56, Cor. 1.4]).

**(5.1.8) Theorem.** *(Michael) Let $M$ be a space metrizable by a complete metric, let $Z$ be a paracompact Hausdorff zero-dimensional space and let $q : Z \to \exp(M)$ be a lower semicontinuous map. Then there exists a continuous selection for $q$.*

**(5.1.9) Corollary.** *Let $M$ be a complete metric space, $Z$ a zero-dimensional paracompact Hausdorff space and $p : M \to Z$ a continuous open and onto map. Then there exists a continuous map $s : Z \to M$ satisfying $p \circ s = Id_Z$.*

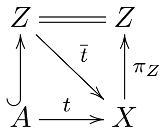
*In particular, if $H$ is a complete metrizable topological abelian group and $G$ is a complete metrizable zero-dimensional topological abelian group then every extension of the form $0 \to H \to X \to G \to 0$ admits a continuous cross-section.*

*Proof.* For the first part consider the map $q : Z \to \exp(M)$ defined by $q(x) = p^{-1}(x)$. In view of (5.1.7.$*$) $q$ is lower semicontinuous and by (5.1.8) there exists a continuous selection for $q$ which will be the desired continuous map $s : Z \to M$.

For the second part, just notice that since completeness and metrizability are three space properties, $X$ is complete and metrizable. Metrizable spaces are paracompact, hence in virtue of the first part of the proof $\pi$ admits a continuous cross-section.

■

**(5.1.10) Corollary.** *Let $X$ be a closed subspace of the product $Z \times M$ where $M$ is a space metrizable by a complete metric and $Z$ is a paracompact Hausdorff zero-dimensional space. Assume also that the canonical projection $\pi_Z : X \to Z$; $(z, m) \mapsto z$ is open and onto and that $A \subset Z$ is closed. Then every continuous function $t : A \to X$ such that $\pi_Z \circ t = Id_A$ can be extended to a continuous function $\bar{t} : Z \to X$ also satisfying $\pi \circ \bar{t} = Id_Z$, in* *other words there exists a commutative diagram:*




*Proof.* Let us start with the following claim:
*(∗) Consider the projection $\pi_M : X \to M$; $(z, m) \mapsto m$. Then the following map is lower semicontinuous:*

$$\begin{array}{rccc} p: & Z & \longrightarrow & \exp(M) \\ & z & \longmapsto & \left\{ \begin{array}{l} \pi_M(\pi_Z^{-1}(z)) \text{ if } z \notin A \\ \{\pi_M \circ t(z)\} \text{ if } z \in A \end{array} \right. \end{array}$$

We need to verify that for every $V$ open in $M$ the set $V_p = \{z \in Z : p(z) \cap V \neq \emptyset\} \subset Z$ is open. $\pi_Z$ is open and onto, hence according to (5.1.7.∗) the map $q : Z \to \exp(X)$; $z \mapsto \pi_Z^{-1}(z)$ is lower semicontinuous. Notice that

$$\begin{aligned} V_p = \{z \notin A : \pi_M(\pi_Z^{-1}(z)) \cap V \neq \emptyset\} &\cup \{z \in A : \pi_M(t(z)) \cap V \neq \emptyset\} \\ &= \{z \notin A : \pi_Z^{-1}(z) \cap \pi_M^{-1}(V) \neq \emptyset\} \cup (\pi_M \circ t)^{-1}(V) \\ &= \Big(\big\{z \in Z : q(z) \cap \pi_M^{-1}(V) \neq \emptyset\big\} \cap (Z \backslash A)\Big) \cup (\pi_M \circ t)^{-1}(V) \\ &= \Big(\big(\pi_M^{-1}(V)\big)_q \cap (Z \backslash A)\Big) \cup (\pi_M \circ t)^{-1}(V). \end{aligned}$$

Since $\pi_M, t$ are continuous, $Z \backslash A, V$ are open and $q$ is lower semicontinuous we deduce that, being the union of two open sets, $V_p$ is open. □

In view of (∗), we deduce, using (5.1.8) that $p$ admits a continuous selection $S : Z \to M$. Notice that if $z \notin A$, $S(z) \in \pi_M(\pi_Z^{-1}(z))$ and $S(z) = \pi_M(x_0)$ for some $x_0 \in \pi_Z^{-1}(z)$ which means that $x_0 = (z, S(z)) \in X$. Otherwise if $z \in A$ $(z, S(z)) = (z, \pi_M \circ t(z)) = t(z) \in X$, therefore the map $\bar{t} : Z \to X$; $z \mapsto (z, S(z))$ is well-defined. Since $\pi_Z(\bar{t}(z)) = \pi_Z(z, S(z)) = z$, and $\bar{t}_{|A} = t$, $\bar{t}$ is the desired map. ■

**(5.1.11) Lemma.** *If a topological abelian group $G$ has countable pseudocharacter then it admits a coarser metrizable group topology.*

*Proof.* Say that $\{0\} = \bigcap_{n<\omega} O_n$ with $O_n$ open for all $n < \omega$, by [AT08, 3.4.18] we can consider for each $O_n$ a metrizable topological abelian group $H_n$ and a continuous epimorphism $\pi_n : G \to H_n$ with

$$\pi_n^{-1}(V_n) \subset O_n \tag{E36}$$

for some $V_n \in \mathcal{N}_0(H_n)$. The diagonal product $\pi = \triangle_{n\in\omega}\pi_n : G \to \prod_{n\in\omega} H_n$ is, in virtue of (E36), a continuous monomorphism. Since $\pi(G)$ is metrizable, we can consider in $G$ the metrizable group topology induced by $\pi$, which will be coarser than the original topology. ■

**(5.1.12) Lemma.** *If $N$ is an admissible subgroup of a topological abelian group $X$, then $X/N$ admits a coarser metrizable group topology.*

*Proof.* Consider a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods 0 in $X$ such that $U_{n+1}+U_{n+1}+U_{n+1} \subset U_n$ for each $n \in \omega$ and $N = \bigcap_{n\in\omega} U_n$ (see (3.5.4)). Taking $\pi\colon X \to X/N;\ x \mapsto x+N$, $\pi^{-1}\pi(U_{n+1}) = U_{n+1}+N \subset U_{n+1}+U_{n+1} \subset U_n$, for each $n \in \omega$. Using this,

$$\pi^{-1}\left(\bigcap_{n\in\omega} \pi(U_{n+1})\right) \subset \bigcap_{n\in\omega} U_n = N,$$

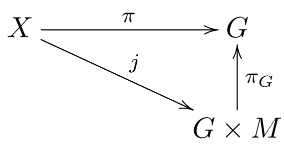
i.e. the set $\bigcap_{n\in\omega} \pi(U_{n+1})$ contains only the neutral element $0+N$ and $X/N$ has countable pseudocharacter. According to (5.1.11), $X/N$ admits a coarser metrizable group topology. ■

The second part of the following lemma will be generalized in (5.1.14) where we will drop the metrizability restriction on the group $K$.

**(5.1.13) Lemma.** *Let $G, X$ be topological abelian groups and let $\pi : X \to G$ be an open continuous epimorphism. Suppose that* $\ker\pi = K$ *is compact and metrizable and that $Y \subset G$ is a zero-dimensional $k_\omega$-space, then there exists a continuous map $s\colon Y \to X$ satisfying $\pi \circ s = \mathrm{Id}_Y$.*

*Proof.* Let us start proving the following claim:

*Claim 1. There exists a metrizable abelian group $M$ and an isomorphic topological embedding $j : X \to G \times M$ making the following diagram commutative,*



(E37)

*where $\pi_G : G \times M \to G$ is the natural projection.*

Since the group $K$ is metrizable, it has a countable local base at 0, say, $\{V_n : n \in \omega\}$. Then there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the neutral element 0 in $X$ such that $U_{n+1}+U_{n+1}+U_{n+1} \subset U_n$ and $U_n \cap K \subset V_n$, for each $n \in \omega$. Clearly $N = \bigcap_{n\in\omega} U_n$ is an admissible subgroup of $X$ and $N \cap K = \bigcap_{n\in\omega} U_n \cap K \subset \bigcap_{n<\omega} V_n = \{0\}$. By (5.1.12), the quotient group $X/N$ admits a coarser metrizable topological group topology $\mathcal{T}$. We denote the topological group $(X/N, \mathcal{T})$ by $M$. Clearly, the quotient map $p : X \to M$ defined by $p(x) = x + M$ is continuous since $\mathcal{T}$ is coarser than the original topology. Consider the diagonal product $j = \pi \vartriangle p : X \to G \times M$.

Since $K$ is compact, $\pi$ is perfect and according to (1.2.4.ii) $j$ is perfect. It is easy to see that $j$ is one-to-one. Indeed, take an arbitrary element $x \in X$

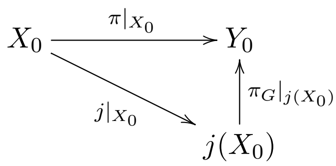
distinct from 0. If $x \notin K$, then $\pi(x) \neq 0$ and hence $j(x) \neq 0$. If $x \in K$, then $x \notin N$, whence it follows that $p(x) \neq 0$ and $j(x) \neq 0$. Thus $j$ is a perfect one-to-one homomorphism of $X$ onto $j(X)$, and so $j$ is a topological isomorphism of $X$ onto the subgroup $j(X)$ of $G \times M$. The commutativity of (E37) follows from the construction of $j$. □

As $Y$ is a $k_\omega$-space there exists an increasing sequence $\{Y_n : n \in \omega\}$ of compact subspaces that determines its topology (1.2.6). Let $X_n = \pi^{-1}(Y_n)$ for all $n < \omega$.

*Claim 2. There exist a family of continuous mappings $\{t_n : Y_n \to j(X_n) : n < \omega\}$ such that $t_{n+1|Y_n} = t_n$ and $\pi_G \circ t_n = Id_{Y_n}$.*

We construct the required family $\{t_n : Y_n \to j(X_n) : n < \omega\}$ by induction. Clearly $Y_0$ is a compact zero-dimensional subspace of $Y$. Since $p$ is perfect, $X_0 = p^{-1}(Y_0)$ is a compact subspace of $X$ and $K_0 = p_M(j(X_0))$ is a compact subspace of $M$. Thus $j(X_0)$ is a compact subspace of $Y_0 \times K_0$, where $K_0$ is a compact metrizable space.

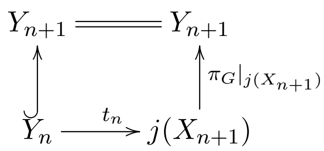
The restriction of $\pi$ to $X_0$ is a continuous open map of $X_0$ onto $Y_0$ and, therefore, the restriction of $\pi_G$ to $j(X_0)$ is a continuous open map of $j(X_0)$ onto $Y_0$. We obtain the following commutative diagram:




(E38)

By (5.1.10) (with $A = \emptyset$, $Z = Y_0$) there exists a continuous map $t_0 \colon Y_0 \to j(X_0)$ such that $\pi_G \circ t_0 = \mathrm{Id}_{Y_0}$.

Suppose that for some $n \in \omega$, we have defined a continuous map $t_n \colon Y_n \to j(X_n)$ satisfying $\pi_G \circ t_n = \mathrm{Id}_{Y_n}$. Note that $X_{n+1} = \pi^{-1}(Y_{n+1})$ and $j(X_{n+1})$ are compact subspaces of $X$ and $G \times M$, respectively. Hence $K_{n+1} = p_M(j(X_{n+1}))$ is a compact subspace of $M$ and $j(X_{n+1}) \subset Y_{n+1} \times K_{n+1}$. We obtain the following commutative diagram




(E39)

By (5.1.10) (this time with $A = Y_n$ and $t = t_n$), there exists a continuous map $t_{n+1}$ of $Y_{n+1}$ to $j(X_{n+1})$ which extends $t_n$ and satisfies $\pi_G \circ t_{n+1} = \mathrm{Id}_{Y_{n+1}}$. This proves Claim 2. □

Consider now for each $n < \omega$ the map $s_n = j^{-1} \circ t_n : Y_n \to X_n$. Because of the way in which the family $\{t_n : n < \omega\}$ was constructed and the commutativity of (E37), for every $n < \omega$ $s_{n+1|Y_n} = s_n$ and $\pi \circ s_n = \mathrm{Id}_{Y_n}$. Let $s$ be the continuous map of $Y$ to $X$ which coincides with $s_n$ on $Y_n$ for each $n \in \omega$ (see (1.2.6))). It is clear from the construction that $\pi \circ s = \mathrm{Id}_Y$, and the proof is complete. ■

**(5.1.14) Theorem.** *Let $G, X$ be topological abelian groups and let $\pi : X \to G$ be an open continuous epimorphism. Suppose that $\ker \pi = K$ is compact and that $Y \subset G$ is a zero-dimensional and $k_\omega$-space; then there exists a continuous map $s\colon Y \to X$ satisfying $\pi \circ s = \mathrm{Id}_Y$. In particular, if $G$ is a zero-dimensional $k_\omega$-space, there exists a continuous cross-section for $\pi$.*

*Proof.* Let $\tau$ be the character of $X$, i.e. the minimum cardinality of a local base at the identity element of $X$. Using (3.5.4.i) we can construct a family $\{N_\alpha : \alpha < \tau\}$ of admissible subgroups of $X$ such that:
*$(*)$ every neighborhood of the neutral element $0$ in $X$ contains $N_\alpha$, for some $\alpha < \tau$.*

For every $\alpha < \tau$, take $\varphi_\alpha : X \to X/N_\alpha$; $x \mapsto x + N_\alpha$. Put $\pi_0 = \pi$, and for any $0 < \alpha \leq \tau$ define the diagonal product

$$\begin{array}{rccc} \pi_\alpha = \pi \triangle \big(\triangle_{\gamma<\alpha}\varphi_\gamma\big) : & X & \longrightarrow & X/K \times \big(\prod_{\gamma<\alpha} X/N_\gamma\big) \\ & x & \longmapsto & \Big(\pi(x), \big(\varphi_\gamma(x)\big)_{\gamma<\alpha}\Big) \end{array}$$

Call $X_\alpha$ the subgroup $\pi_\alpha(X)$ of $X/K \times (\prod_{\gamma<\alpha} X/N_\gamma)$. Given ordinals $\alpha, \beta$ with $\beta < \alpha < \tau$, consider

$$\begin{array}{lccc} \pi_{\alpha,\beta} : & X_\alpha & \longrightarrow & X_\beta \\ & \pi_\alpha(x) = \Big(\pi(x), \big(\varphi_\gamma(x)\big)_{\gamma<\alpha}\Big) & \longmapsto & \pi_\beta(x) = \Big(\pi(x), \big(\varphi_\gamma(x)\big)_{\gamma<\beta}\Big) \end{array}$$

It is clear that $\pi_{\alpha,\beta}$ is continuous and satisfies $\pi_\beta = \pi_{\alpha,\beta} \circ \pi_\alpha$. Then $\mathcal{P} = \{X_\alpha, \pi_{\alpha,\beta} : \beta < \alpha < \tau\}$ is an inverse system of topological abelian groups.
*Claim. For each limit ordinal $\alpha < \tau$, consider the inverse system $\mathcal{P}_\alpha = \{X_\gamma,\, \pi_{\gamma,\beta} : \beta < \gamma < \alpha\}$. The following maps are topological isomorphisms*

$$\begin{array}{cccc} \Phi : & X & \longrightarrow & \lim_{\leftarrow} \mathcal{P} \\ & x & \longrightarrow & \big(\pi_\gamma(x)\big)_{\gamma<\tau} \end{array} \qquad \begin{array}{cccc} \Phi_\alpha : & X_\alpha & \longrightarrow & \lim_{\leftarrow} \mathcal{P}_\alpha \\ & \pi_\alpha(x) & \longrightarrow & \big(\pi_\gamma(x)\big)_{\gamma<\alpha} \end{array}$$

Indeed, since $\pi_0 = \pi$, $\Phi$ is a perfect map (1.2.4.ii). By $(*)$, $\Phi$ is one-to-one. So to prove that $\Phi$ is a homeomorphism it suffices to check that it is onto. Pick a basic open set $W = (V_\beta \times \prod_{\gamma\neq\beta} X_\gamma) \cap \lim_{\leftarrow} \mathcal{P}$ of $\lim_{\leftarrow} \mathcal{P}$ with $V_\beta$ open in $X_\beta$ (see ( 1.2.5)). Taking $y \in \pi_\beta^{-1}(V_\beta)$ we obtain that $\Phi(y) \in W$ and therefore $\Phi(X)$ is dense in $\lim_{\leftarrow} \mathcal{P}$. Since $\Phi$ is perfect, $\Phi(X)$ is closed in $\lim_{\leftarrow} \mathcal{P}$ and consequently $\Phi$ is onto.

The same argument can be used for $\Phi_\alpha$ once we show that $\pi_{\alpha,0} : X_\alpha \to X_0 = X/K$ is a perfect map. It is clear that $\pi_{\alpha,0}$ is onto and continuous.

Let us see that it is open: Fix an open set $V$ in $X_\alpha$. We have $\pi_{\alpha,0}(V) = \pi_{\alpha,0}(\pi_\alpha(\pi_\alpha^{-1}(V))) = \pi_0(\pi_\alpha^{-1}(V))$ which is open since $\pi_0$ is open and $\pi_\alpha$ is continuous. We deduce that $\pi_{\alpha,0}$ is a quotient map. Using a similar argument we prove that $\ker \pi_{\alpha,0} = \pi_\alpha(K)$. As a quotient map with compact kernel, $\pi_{\alpha,0}$ is a perfect map. $\square$

We are going to define a system of continuous maps $s_\alpha\colon Y \to X_\alpha$ satisfying the following condition for all $\alpha, \beta$ with $0 \leq \beta < \alpha < \tau$:

*(**)* $\pi_{\alpha,\beta} \circ s_\alpha = s_\beta$.

We start letting $s_0 = \mathrm{Id}_Y$. Suppose that the system $\{s_\beta : \beta < \alpha\}$ satisfying $(**)$ is defined for some ordinal $\alpha$ with $0 < \alpha < \tau$. If $\alpha$ is limit, consider $\Phi_\alpha$ the topological isomorphism constructed in the Claim and define $s_\alpha$ as

$$\begin{array}{ccccc} & & \xrightarrow{\;s_\alpha=\Phi_\alpha^{-1}\circ\mathcal{S}_\alpha\;} & & \\ Y & \xrightarrow{\;\mathcal{S}_\alpha\;} & \lim_{\leftarrow} \mathcal{P}_\alpha & \xrightarrow{\;\Phi_\alpha^{-1}\;} & X_\alpha \\ y & \longmapsto & (s_\beta(y))_{\beta<\alpha} & & \\ & & (\pi_\beta(x))_{\beta<\alpha} & \longmapsto & \pi_\alpha(x) \end{array}$$

Notice that $\mathcal{S}_\alpha : Y \to \lim_{\leftarrow} \mathcal{P}_\alpha;\ y \mapsto (s_\beta(y))_{\beta<\alpha}$ is a well-defined continuous map because of $(**)$ (see (1.2.5)). An easy verification shows that $\pi_{\alpha,\beta}\circ s_\alpha = s_\beta\ \forall \beta < \alpha$.

Suppose now that $\alpha$ is a successor ordinal, say, $\alpha = \nu + 1$. Notice that $\ker \pi_\alpha = K \cap (\bigcap_{\beta<\alpha} N_\beta)$ is a compact group and

$$\begin{aligned} \ker \pi_{\alpha+1,\alpha} &= \big\{\big(\pi_\alpha(x), \varphi_\alpha(x)\big) : \pi_\alpha(x) = 0\big\} \cong \big\{\varphi_\alpha(x) : x \in \ker \pi_\alpha\big\} \\ &= \varphi_\alpha(\ker \pi_\alpha) \cong \ker \pi_\alpha/(\ker \pi_\alpha \cap N_\alpha) \end{aligned}$$

The group $\ker \pi_\alpha/(\ker \pi_\alpha \cap N_\alpha)$ is also metrizable because it is a compact space with countable pseudocharacter. Hence the homomorphism $\pi_{\alpha+1,\alpha}$ satisfies the hypothesis of (5.1.13) . It follows from $\pi_{\nu,0} \circ s_\nu = \mathrm{Id}_Y$ that $s_\nu$ is a homeomorphism of $Y$ onto a subspace of $X_\nu$. In particular, $Y_\nu = s_\nu(Y)$ is a zero-dimensional $k_\omega$-subspace of $X_\nu$. Applying (5.1.13) to the open homomorphism $\pi_{\nu+1,\nu} = \pi_{\alpha,\nu}$, we deduce that there exists a continuous map $t_\nu\colon Y_\nu \to X_\alpha$ such that $\pi_{\alpha,\nu} \circ t_\nu = \mathrm{Id}_{Y_\nu}$. Let us put $s_\alpha = t_\nu \circ s_\nu$. It is clear that the system $\{s_\beta : 0 \leq \beta \leq \alpha\}$ satisfies $(**)$. This finishes our recursive construction.

To conclude the proof, consider the continuous map

$$\begin{array}{rccc} s = \Phi^{-1} \circ \big(\triangle_{\alpha<\tau} s_\alpha\big) : & Y & \longrightarrow & X \\ & y & \longmapsto & \Phi^{-1}\Big(\big(s_\alpha(y)\big)_{\alpha<\tau}\Big) \end{array}$$

Since $\pi\circ\Phi^{-1}\Big(\big(\pi_\alpha(x)\big)_{\alpha<\tau}\Big) = \pi\circ\Phi^{-1}\Big(\big(\pi(x), (\phi_\alpha(x))_{\alpha<\tau}\big)\Big) = \pi(x)$, it follows that

$$\pi \circ s(y) = \pi \circ \Phi^{-1}\Big(\big(s_\alpha(y)\big)_{\alpha<\tau}\Big) = s_0(y) = \mathrm{id}_Y(y).$$

■

**(5.1.15) Corollary.** *Let $K$ be a compact abelian group and $A(Y)$ the free abelian topological group on a zero-dimensional $k_\omega$-space $Y$. Then* $\mathrm{Ext}(A(Y), K) = 0$.

*Proof.* Let $E : 0 \to K \xrightarrow{\imath} X \xrightarrow{p} A(Y) \to 0$ be an extension of topological groups. By (5.1.14), there exists a continuous map $s\colon Y \to X$ satisfying $p \circ s = \mathrm{Id}_Y$. Since $A(Y)$ is the free abelian topological group over $Y$, the map $s$ extends to a continuous homomorphism $S\colon A(Y) \to X$; $\sum_{i\le k} n_i y_i \mapsto \sum_{i\le k} n_i s(y_i)$ (see (1.3.14)). For every $\sum_{i\le k} n_i y_i \in A(Y)$,

$$p \circ S\Big(\sum_{i\le k} n_i y_i\Big) = \sum_{i\le k} n_i p(s(y_i)) = \sum_{i\le k} n_i y_i.$$

Hence (2.1.7) implies that $E$ splits. ■

**(5.1.16) Corollary.** *Let $H$ and $G$ be topological abelian groups. If $H$ is compact and $G$ is a zero-dimensional $k_\omega$-space, then every extension of the form $0 \to H \to X \to G \to 0$ admits a continuous cross-section.*

## §5.2 Topological vector spaces and continuous cross-sections

We proceed now to prove (5.2.2), in which we will obtain continuous cross-sections for extensions of topological abelian groups of the form $0 \to Y \to X \to Z \to 0$ where $Y$ is a Fréchet topological vector space and $Z$ is a metrizable topological vector space. The key part of the proof of (5.2.2) lies in (5.2.1) which is due to Michael and whose proof can be found in [BP75, Prop. 7.1 of Chap. II].

**(5.2.1) Proposition.** *Let $X$ and $Z$ be complete metrizable topological vector spaces and let $\pi : X \to Z$ be an onto continuous linear mapping such that* $\ker \pi$ *is a Fréchet topological vector space. Then $\pi$ admits a continuous cross-section.*

**(5.2.2) Theorem.** *If $Y$ is a Fréchet topological vector space and $Z$ is a metrizable topological vector space then every extension of topological abelian groups of the form $E : 0 \to Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \to 0$ admits a continuous cross-section.*

*Proof.* By (3.2.3) we can consider the completed extension $\varrho E : 0 \to Y \stackrel{\varrho\imath}{\to} \varrho X \stackrel{\varrho\pi}{\to} \varrho Z \to 0$ obtaining the following commutative diagram:

$$\begin{array}{lccccccccc}
\varrho E : & 0 & \longrightarrow & Y & \xrightarrow{\varrho\imath} & \varrho X & \xrightarrow{\varrho\pi} & \varrho Z & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow & & \\
E : & 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0
\end{array}$$

Applying (4.2.2) we deduce that $X$ admits a compatible topological vector space structure such that $E$ can be regarded as an extension of topological vector spaces. By (5.2.1) there exists a continuous cross-section $S : \varrho Z \to \varrho X$ for $\varrho\pi$. Using (5.1.5) we obtain a continuous map $R : \varrho X \to Y$ such that $R(\imath(y)+x) = R(x)+y$ and $R(\imath(y)) = y \ \forall y \in Y, x \in X$. The restriction $r = R_{|X} : X \to H$ is a continuous map that has the same properties, hence (5.1.5) guarantees the existence of the desired cross-section for $\pi$. ■

**(5.2.3) Notes.** (5.1.6) is [Cab03, Lemma 11], the proof is taken from [BCD13, Prop. 31]. The proof of (5.1.10) uses the same argument as [AT08, Lemma 4.1.4]. (5.1.11) is [AT08, Coro. 3.4.26]. (5.1.13) and (5.1.14) are [BCDT16, Lemma 2.5] and [BCDT16, Th. 2.8] respectively.

# Chapter 6

# Quasi-homomorphisms

Our aim in this chapter is to develop the theory of quasi-homomorphisms, which were defined by Cabello in [Cab03]. We will start with §6.1 and §6.2, where we will study the main properties of quasi-homomorphisms. In §6.3 we will show how to construct extensions of topological abelian groups from quasi-homomorphisms and we will find situations in which quasi-homomorphisms characterize the Ext group.

We will finish the chapter with §6.4 where we will focus on the quasi-homomorphisms from a topological abelian group to $\mathbb{R}$ or $\mathbb{T}$. This final section will lead us to the following chapter, in which we will combine the techniques developed here with the ones in the previous chapters to study the extensions of the form $0 \to \mathbb{R} \to X \to G \to 0$ and $0 \to \mathbb{T} \to X \to G \to 0$.

## §6.1 Quasi-homomorphisms and pseudo-homomorphisms

**(6.1.1) Definition.** Let $G$ and $H$ be topological abelian groups. A map $q : G \to H$ that has the properties:

(a) $q(0) = 0$,

(b) $\Delta_q : G \times G \to H;\ (x, y) \mapsto q(x+y) - q(x) - q(y)$ is continuous at $(0, 0)$

is called a *quasi-homomorphism*. If the quasi-homomorphism $q$ also satisfies that $\Delta_q$ is continuous then we will say that it is a *pseudo-homomorphism*.

**(6.1.2) Lemma.** *Let $G$ and $H$ be topological groups and let $q : G \to H$ be a quasi-homomorphism. Then the following are equivalent*

*(i) $q$ is a pseudo-homomorphism.*

*(ii) For every net $\{x_\sigma : \sigma \in \Sigma\} \subset G$ with $\lim_{\sigma\in\Sigma} x_\sigma = x$, one has that $\lim_{\sigma\in\Sigma}(q(x - x_\sigma) - q(x) + q(x_\sigma)) = 0$ in $H$.*

*Proof.* (i)⇒ (ii). Let $\{x_\sigma : \sigma \in \Sigma\}$ be a net in $G$ with $\lim_{\sigma\in\Sigma} x_\sigma = x$. From the continuity of $\Delta_q$ it follows that $\lim_{\sigma\in\Sigma} \Delta_q(x_\sigma, -x_\sigma) = \Delta_q(x, -x)$, which means that $\lim_{\sigma\in\Sigma}(q(x_\sigma - x_\sigma) - q(x_\sigma) - q(-x_\sigma)) = q(x - x) - q(x) - q(-x)$. Consequently,

$$\lim_{\sigma\in\Sigma}(q(x_\sigma) + q(-x_\sigma)) = q(x) + q(-x). \tag{E40}$$

Since $\lim_{\sigma\in\Sigma} \Delta_q(x, -x_\sigma) = \Delta_q(x, -x)$, we see that $\lim_{\sigma\in\Sigma}(q(x-x_\sigma)-q(x)-q(-x_\sigma)) = q(x - x) - q(x) - q(-x)$ and

$$\lim_{\sigma\in\Sigma}(q(x - x_\sigma) - q(-x_\sigma)) = -q(-x). \tag{E41}$$

Applying (E40) and (E41) we deduce that

$$\begin{aligned}
&\lim_{\sigma\in\Sigma}(q(x - x_\sigma) - q(x) + q(x_\sigma)) \\
&\qquad = \lim_{\sigma\in\Sigma}\Big(\big(q(x - x_\sigma) - q(-x_\sigma)\big) + \big(q(-x_\sigma) + q(x_\sigma)\big) - q(x)\Big) \\
&\qquad = -q(-x) + \big(q(x) + q(-x)\big) - q(x) = 0.
\end{aligned}$$

(ii)⇒ (i). Pick two nets $\{x_\sigma : \sigma \in \Sigma\}, \{y_\sigma : \sigma \in \Sigma\} \subset G$ with $\lim_{\sigma\in\Sigma} x_\sigma = x$ and $\lim_{\sigma\in\Sigma} y_\sigma = y$. Notice that

$$\lim_{\sigma\in\Sigma} \big(q(x + y - x_\sigma - y_\sigma) - q(x + y) + q(x_\sigma + y_\sigma)\big) = 0, \tag{E42}$$

$$\lim_{\sigma\in\Sigma} \big(q(x - x_\sigma) - q(x) + q(x_\sigma)\big) = 0, \tag{E43}$$

$$\lim_{\sigma\in\Sigma} \big(q(y - y_\sigma) - q(y) + q(y_\sigma)\big) = 0, \tag{E44}$$

$$\lim_{\sigma\in\Sigma} \big(q(x + y - x_\sigma - y_\sigma) - q(x - x_\sigma) - q(y - y_\sigma)\big) = 0. \tag{E45}$$

Combining (E42) and (E45) we obtain that

$$\lim_{\sigma\in\Sigma} \big(q(x + y) - q(x_\sigma + y_\sigma) - q(x - x_\sigma) - q(y - y_\sigma)\big) = 0. \tag{E46}$$

Adding (E46), (E43) and (E44) we deduce that:

$$\lim_{\sigma\in\Sigma} \Big( - q(x_\sigma + y_\sigma) + q(x_\sigma) + q(y_\sigma) + \big(q(x + y) - q(x) - q(y)\big)\Big) = 0, \tag{E47}$$

and this implies the continuity of $\Delta_q$. ■

## §6.2 The group topology defined by a quasi-homomorphism

**(6.2.1) Proposition-definition.** *Given a quasi-homomorphism $q : G \to H$, for every $V$ and $U$ open neighborhoods of $0$ in $H$ and $G$ respectively, consider the subset*

$$W(V, U) = \{(h, g) \in H \times G : g \in U, h \in q(g) + V\} \subset H \times G.$$

*The family $\mathcal{N}$ of all sets of the form $W(V,U)$ is a system of open neighborhoods of zero in $H \times G$ for a group topology. The abelian group $H \times G$ endowed with the group topology defined by $\mathcal{N}$ is denoted by $H \oplus_q G$.*

*Proof.* We need to prove that $\mathcal{N}$ satisfies the following properties (see (1.3.1)):

(a) For every $W(V,U), W(V',U') \in \mathcal{N}$ there exists another $W(V'',U'') \in \mathcal{N}$ with $W(V'',U'') \subset W(V,U) \cap W(V',U')$.

(b) For every $W(V,U) \in \mathcal{N}$, there exists $W(V',U') \in \mathcal{N}$ with $-W(V',U') \subset W(V,U)$.

(c) For every $W(V,U) \in \mathcal{N}$, there exists $W(V',U') \in \mathcal{N}$ with $W(V',U') + W(V',U') \subset W(V,U)$.

(d) For every $W(V,U) \in \mathcal{N}$ and $(h,g) \in W(V,U)$, there exists $W(V',U') \in \mathcal{N}$ with $(h,g) + W(V',U') \subset W(V,U)$.

(a). For every $W(V,U), W(V',U') \in \mathcal{N}$, $W(V \cap V', U \cap U') \subset W(V,U) \cap W(V',U')$.

(b). Fix an open neighborhood $V'$ of 0 in $H$ with $V' - V' \subset V$. Choose also an open neighborhood $U'$ of 0 in $G$ with $-U' \subset U$ and $\Delta_q(U' \times -U') \subset V'$. Note that

$$\begin{aligned} g \in U' \quad &\Rightarrow (g,-g) \in U' \times -U' \\ &\Rightarrow q(g-g) - q(g) - q(-g) \in \Delta_q(U' \times -U') \subset V' \\ &\Rightarrow -q(g) \in q(-g) + V' \end{aligned} \tag{E48}$$

Pick $(h,g) \in W(V',U')$ and let us check that $(-h,-g) \in W(V,U)$. In view of (E48), since $h \in q(g) + V'$,

$$-h \in -q(g) - V' \subset q(-g) + V' - V' \subset q(-g) + V.$$

(c). Let $U'$ and $V'$ be open neighborhoods of 0 in $G$ and $H$ respectively with the following properties:

$$\begin{aligned} U' + U' &\subset U \\ V' + V' + V' &\subset V \\ \Delta_q(U' \times U') &\subset V' \end{aligned}$$

For every $(g_1,h_1)$, $(g_2,h_2) \in W(V',U')$, $g_1 + g_2 \in U' + U' \subset U$ and $h_1 + h_2 \in q(g_1) + q(g_2) + V' + V' \subset q(g_1+g_2) + V' + V' + V' \subset q(g_1+g_2) + V$.

(d). Choose an open neighborhood $V'$ of 0 in $H$ with $h + V' \subset V$ and $U'$ an open neighborhood of 0 in $G$ with $g + U' \subset U$. If $(h',g') \in W(V',U')$, $g + g' \in g + U' \subset U$ and $h + h' \in h + V' \subset V$ thereby $(h+h', g+g') \in W(V,U)$. ∎

**(6.2.2) Proposition.** *Let $H$, $G$ be topological abelian groups and let $q : G \to H$ be a quasi-homomorphism. For every net $\{(h_\sigma, x_\sigma) : \sigma \in \Sigma\} \subset H \oplus_q G$,*

$$\lim_{\sigma\in\Sigma}(h_\sigma, x_\sigma) = 0 \Longleftrightarrow \begin{cases} \lim_{\sigma\in\Sigma} x_\sigma = 0 \\ \lim_{\sigma\in\Sigma}(h_\sigma - q(x_\sigma)) = 0 \end{cases}$$

*Proof.* Suppose that $\lim_{\sigma\in\Sigma}(h_\sigma, x_\sigma) = 0$. To prove that $\lim_{\sigma\in\Sigma} x_\sigma = 0$, fix $U$ an arbitrary neighborhood of 0 in $G$. Pick any $N \in \mathcal{N}_0(H)$, by assumption there exists $\sigma_0 \in \Sigma$ with $(x_\sigma, h_\sigma) \in W(N, U)\ \forall\sigma \geq \sigma_0$ which implies that $x_\sigma \in U\ \forall\sigma \geq \sigma_0$. To check that $\lim_{\sigma\in\Sigma}(h_\sigma - q(x_\sigma)) = 0$, fix an arbitrary neighborhood $V$ of 0 in $H$ and choose any $M \in \mathcal{N}_0(G)$. Taking $\sigma_1 \in \Sigma$ such that $(h_\sigma, x_\sigma) \in W(V, M)\ \forall\sigma \geq \sigma_1$ one finds that $(h_\sigma - q(x_\sigma)) \in V\ \forall\sigma \geq \sigma_1$.

Conversely, suppose that $\lim_{\sigma\in\Sigma} x_\sigma = 0$ and $\lim_{\sigma\in\Sigma}(h_\sigma - q(x_\sigma)) = 0$. Let $W(V, U)$ be an arbitrary neighborhood of 0 in $H \oplus_q G$. By definition of a convergent net, there exist $\sigma_0', \sigma_0 \in \Sigma$ such that $x_\sigma \in U\ \forall\sigma \geq \sigma_0$ and $(h_\sigma - q(x_\sigma)) \in V\ \forall\sigma \geq \sigma_0'$. Finally, choosing $\sigma_0'' \in \Sigma$ such that $\sigma_0'' \geq \sigma_0$ and $\sigma_0'' \geq \sigma_0'$, $(h_\sigma, x_\sigma) \in W(V, U)\ \forall\sigma \geq \sigma_0''$. ■

## §6.3 Quasi-homomorphisms and the Ext group

**(6.3.1) Extensions defined by quasi-homomorphisms.** Given a quasi-homomorphism $q : G \to H$ between topological abelian groups, consider $\imath_H : H \to H \oplus_q G$; $h \mapsto (h, 0)$ and $\pi_G : H \oplus_q G \to G$; $(h, g) \mapsto g$.

*(i) The sequence $E_q : 0 \to H \overset{\imath_H}{\to} H \oplus_q G \overset{\pi_G}{\to} G \to 0$ is an extension of topological abelian groups.*

*Proof.* The fact that $\imath_H$ is continuous and open onto its image follows from the identity $W(V, U) \cap \imath_H(H) = \imath_H(V)$, which is true for every $U \in \mathcal{N}_0(G)$ and every $V \in \mathcal{N}_0(H)$. $\pi_G$ is continuous and open because $\pi_G(W(V, U)) = U$ for every $U \in \mathcal{N}_0(G)$ and every $V \in \mathcal{N}_0(H)$ ("$\subset$" is trivial; for "$\supset$", note that $g = \pi_G((q(g), g))$ where $q(g) \in q(g) + V$ for every $g \in U$). □

The sequence $E_q$ is called the *extension defined by the quasi-homomorphism* $q : G \to H$.

Notice that the composition of a quasi-homomorphism with a continuous homomorphism is a quasi-homomorphism.

*(ii) For every continuous homomorphism $\gamma : G' \to G$, $E_q\gamma$ is equivalent to $E_{q\circ\gamma} : 0 \to H \overset{\imath_H}{\to} H \oplus_{q\circ\gamma} G' \overset{\pi_{G'}}{\to} G' \to 0$ Analogously, for every continuous homomorphism $\delta : H \to H'$, $\delta E_q$ is equivalent to $E_{\delta\circ q} : 0 \to H' \overset{\imath_{H'}}{\to} H' \oplus_{\delta\circ q} G \overset{\pi_G}{\to} G \to 0$*

*Proof.* Consider the homomorphism

$$\begin{array}{rccc} \mathrm{Id}_H \times \gamma : & H \oplus_{q\circ\gamma} G' & \longrightarrow & H \oplus_q G \\ & (h, x) & \longmapsto & (h, \gamma(x)) \end{array}$$

If $\{(h_\sigma, x_\sigma), \sigma \in \Sigma\} \subset H \oplus_{q\circ\gamma} G'$ is a net convergent to 0, by (6.2.2), $\lim_{\sigma\in\Sigma} x_\sigma = 0$ in $G$ and $\lim_{\sigma\in\Sigma}(h_\sigma - q(\gamma(x_\sigma))) = 0$ in $H$ which, again by (6.2.2) implies that $\{(h_\sigma, \gamma(x_\sigma)), \sigma \in \Sigma\} \subset H \oplus_q G$ converges to 0. Hence $\mathrm{Id}_H \times \gamma$ is continuous.

$\mathrm{Id}_H \times \gamma$ makes the diagram

$$\begin{array}{llllllllll} E_q: & 0 \longrightarrow & H & \xrightarrow{i_H} & H \oplus_q G & \xrightarrow{\pi_G} & G & \longrightarrow 0 \\ & & \| & & \uparrow{\scriptstyle \mathrm{Id}_H\times\gamma} & & \uparrow{\scriptstyle\gamma} & \\ E_{q\circ\gamma}: & 0 \longrightarrow & H & \xrightarrow{i_H} & H \oplus_{q\circ\gamma} G' & \xrightarrow{\pi_{G'}} & G' & \longrightarrow 0 \end{array} \tag{E49}$$

commutative. Hence, according to (2.2.6), $E_{q\circ\gamma}$ is equivalent to $E_q\gamma$.

For the other case proceed analogously using $\delta\times\mathrm{Id}_G : H\oplus_q G \to H'\oplus_{\delta\circ q} G$ instead of $\mathrm{Id}_H \times \gamma$. $\square$

**(6.3.2) Lemma.** *Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ be an extension of topological groups that splits algebraically. The following are equivalent:*

*(i) $E$ is equivalent to an extension $E_q : 0 \to H \overset{\imath_H}{\to} H \oplus_q G \overset{\pi_G}{\to} G \to 0$ induced by a quasi-homomorphism (resp. pseudo-homomorphism) $q : G \to H$.*

*(ii) There exists a map $r : X \to H$ continuous at the origin (respectively continuous) satisfying $r(\imath(h)) = h$ and $r(x + \imath(h)) = r(x) + h$ for every $h \in H, x \in X$.*

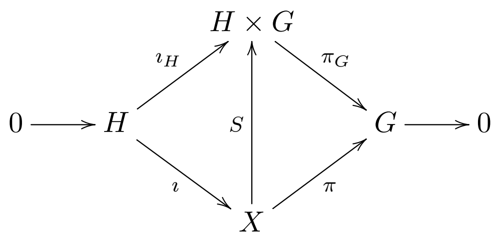
*(iii) $E$ admits a cross-section continuous at 0 (continuous).*

*(iv) There exists a bijection $\phi : H \times G \to X$ continuous at the origin with inverse continuous at the origin (homeomorphism) and such that $\phi(h, 0) = \imath(h)$, $\pi(\phi(h, g)) = g$ for every $h \in H, g \in G$.*

*Proof.* We will only give the details of the proof for the case in which $q$ is a quasi-homomorphism, if $q$ is a pseudo-homomorphism the proof is analogous. By (5.1.5), (ii), (iii) and (iv) are equivalent, it suffices to show that (i) is equivalent to (iii).

(i)⇒(iii). Suppose that $E$ is equivalent to an extension $E_q : 0 \to H \overset{\imath_H}{\to} H \oplus_q G \overset{\pi_G}{\to} G \to 0$ induced by a quasi-homomorphism $q$. Consider $T : X \to H \oplus_q G$ the topological isomorphism witnessing the equivalence between $E$ and $E_q$. Define the map $s : G \to X$ as $s(g) = T^{-1}(q(g), g)$. Using (6.2.2), one can easily show that the map $g \mapsto (q(g), g) \in H \oplus_q G$ is continuous at 0. As $\pi \circ s(g) = \pi \circ T^{-1}(q(g), g) = \pi_G(g) = g$, $s$ is the desired map.

(iii)⇒(i). Consider the trivial extension $E_0 : 0 \to H \overset{\imath_H}{\to} H \times G \overset{\pi_G}{\to} G \to 0$. Since $E$ splits algebraically there exists a homomorphism $S : X \longrightarrow H \times G$ making commutative the diagram




(E50)

Let $\pi_H : H \times G \to H$ be the canonical projection and write $P = \pi_H \circ S$. By the commutativity of (E50), $P \circ \imath = \mathrm{Id}_H$. Let $s : G \to X$ be the cross-section continuous at 0 for $\pi$. We can suppose without loss of generality that $s(0) = 0$. Taking $q = P \circ s$, let us see that $\Delta_q$ is continuous at $(0, 0)$. For every $g_1, g_2 \in G$, we have that $s(g_1 + g_2) - s(g_1) - s(g_2) \in \imath(H)$ (because $\imath(H) = \ker \pi$, and $\pi \circ s = \mathrm{Id}_G$). This means that $P(s(g_1+g_2)-s(g_1)-s(g_2)) = \imath^{-1}(s(g_1+g_2)-s(g_1)-s(g_2))$ and hence, $\Delta_q = \imath^{-1} \circ \Delta_s$. Since $\Delta_s$ is continuous at $(0, 0)$ and $\imath^{-1}$ is continuous, $\Delta_q$ is continuous at $(0, 0)$.

Let us see that the diagonal product $\mathcal{S} = P \triangle \pi : X \to H \oplus_q G$; $x \mapsto (P(x), \pi(x))$ witnesses the equivalence between $E$ and $E_q$. Since $\mathcal{S}$ trivially witnesses the algebraic equivalence, we only need to check that it is a continuous map. Take $\{x_\sigma : \sigma \in \Sigma\}$ a net convergent to 0 in $X$, since $x - s \circ \pi(x) \in \imath(H)$ $\forall x \in X$,

$$P(x_\sigma) - q \circ \pi(x_\sigma) = P(x_\sigma - s \circ \pi(x_\sigma)) = \imath^{-1}(x_\sigma - s \circ \pi(x_\sigma)).$$

Then $\lim_{\sigma \in \Sigma}(P(x_\sigma) - q \circ \pi(x_\sigma)) = 0$ because $\imath^{-1}$ is continuous and $s$ is continuous at 0. In view of (6.2.2), $\lim_{\sigma \in \Sigma} \mathcal{S}(x_\sigma) = 0$ and $\mathcal{S}$ is continuous. ■

**(6.3.3) Approximable quasi-homomorphisms.** A quasi-homomorphism $q : G \to H$ is said to be *approximable* if there exists a homomorphism $a : G \to H$ such that $q - a$ is continuous at 0. In these conditions it is said that *a approximates q*.

*Fact 1. A quasi-homomorphism $q : G \to H$ is approximable if and only if its associated extension $E_q : 0 \to H \overset{\imath_H}{\to} H \oplus_q G \overset{\pi_G}{\to} G \to 0$ splits.*

*Proof.* Suppose that $E_q$ splits. In view of (2.1.7), there exists a continuous homomorphism $P : H \oplus_q G \to H$ such that $P \circ \imath_H = \mathrm{Id}_H$. Define the homomorphism $a : G \to H$ as $a(x) = -P(0, x)$ and let us see that $q - a$ is continuous at zero. Making use of (6.2.2) one can show easily that the map $f : G \to H \oplus_q G$ defined by $f(x) = (q(x), g)$ is continuous at 0, hence the composition $P \circ f$ is also continuous at 0. Notice that

$$P \circ f(x) = P(q(x), x) = P(q(x), 0) + P(0, x) = q(x) + P(0, x) = q(x) - a(x).$$

Therefore, $q - a$ is continuous at 0,

Conversely, assume that there exists a homomorphism $a : G \to H$ that approximates $q$. Define the homomorphism $P : H \oplus_q G \to G$ as $P(h, x) = h - a(x)$. $P$ clearly satisfies $P \circ \imath = \mathrm{id}_H$, let us see that it is continuous at 0. Let $\{(h_\sigma, x_\sigma) : \sigma \in \Sigma\} \subset H \oplus_q G$ a net such that $\lim_{\sigma\in\Sigma}(h_\sigma, x_\sigma) = 0$.

$$P(h_\sigma, x_\sigma) = h_\sigma - a(x_\sigma) = \big(h_\sigma - q(x_\sigma)\big) + \big(q(x_\sigma) - a(x_\sigma)\big).$$

Using (6.2.2), since $\lim_{\sigma\in\Sigma}(q(x_\sigma) - a(x_\sigma)) = 0$ and $\lim_{\sigma\in\Sigma}(h_\sigma - q(x_\sigma)) = 0$, $\lim_{\sigma\in\Sigma}(P(h_\sigma, x_\sigma)) = 0$. □

*Fact 2. A pseudo-homomorphism $q : G \to H$ is approximable if and only if there exists a homomorphism $a : G \to H$ such that $q - a$ is continuous (i.e. pseudo-homomorphisms are approximable if and only if they are "globally" approximable).*

*Proof.* Suppose that $q - a$ is continuous at 0. Pick $x \in G$ with $x \neq 0$ and $\{x_\sigma : \sigma \in \Sigma\} \subset G$ a net with $\lim_{\sigma\in\Sigma} x_\sigma = x$. Using (6.1.2)

$$\begin{aligned}
&\lim_{\sigma\in\Sigma}\big(q(x_\sigma) - a(x_\sigma) - q(x) - a(x)\big)\\
&\quad= \lim_{\sigma\in\Sigma}\Big(\big(q(x_\sigma) - a(x_\sigma) - q(x) - a(x)\big) - \big(q(x - x_\sigma) + q(x) - q(x_\sigma)\big)\Big)\\
&\quad= -\lim_{\sigma\in\Sigma}\big(q(x - x_\sigma) - a(x - x_\sigma)\big)\\
&\quad= 0.
\end{aligned}$$

Conversely, if $q - a$ is continuous, then $\Delta_q$ is also continuous because $\Delta_q = \Delta_{q-a}$. □

**(6.3.4) Examples.** (i) Cabello showed in [Cab03, Th. 1] that a quasi-homomorphism $q : G \to H$ is approximable in any of the following cases:

(a) $G$ is a product of locally compact abelian groups and $H$ is either $\mathbb{R}$ or $\mathbb{T}$.

(b) $G$ is either $\mathbb{R}$ or $\mathbb{T}$ and $H$ is a Banach space.

(ii) Let $A(X)$ be the free abelian topological group on a Tychonoff topological space $X$ and let $H$ any topological abelian group. Then every pseudo-homomorphism $q : A(X) \to H$ is approximable. Indeed, consider $E_q : 0 \to H \overset{\imath_H}{\to} H \oplus_q A(X) \overset{\pi_{A(X)}}{\to} A(X) \to 0$ the extension generated by $q$. According to (6.3.2), $E_q$ admits a continuous cross-section $s : A(X) \to H \oplus_q A(X)$. The restriction $s_{|X} : X \to H \oplus_q A(X)$ is clearly continuous, hence there exists a continuous homomorphism $S : A(X) \to H \oplus_q A(X)$ such that $S_{|X} = s_{|X}$. Accordingly, $\pi \circ S = \mathrm{Id}_{A(X)}$, which implies that $E_q$ splits and $q$ is approximable.

(iii) There are non-approximable pseudo-homomorphisms. It is well-known that there exists a non-splitting extension of topological vector spaces of the form $E : 0 \to \mathbb{R} \to X \to \ell^1 \to 0$ (see (7.2.1) for more detail). Since $\ell^1$ is a complete metrizable topological vector space and $\mathbb{R}$ is a Fréchet space, (5.2.1) tells us that there exists a continuous map $s : \ell^1 \to X$ satisfying $\pi \circ s = \mathrm{Id}_{\ell^1}$. By (6.3.2) $E$ is equivalent to an extension $E_q$ induced by a pseudo-homomorphism $q : \ell^1 \to \mathbb{R}$. Since $E$ does not split $q$ is not approximable.

Since $\mathbb{R}$ is divisible it splits algebraically form $\mathbb{R} \oplus_q \ell^1$ (1.1.5.ii.a). Furthermore, from the fact that $E_q$ is induced by a pseudo-homomorphism follows that $\mathbb{R} \oplus_q \ell^1$ is homeomorphic to $\mathbb{R} \times \ell^1$ (6.3.2.iv). Nevertheless, since $E_q$ does not split, $\mathbb{R}$ does not split topologically from $\mathbb{R} \oplus_q \ell^1$.

**(6.3.5) Definition.** Given two topological abelian groups $G$ and $H$ we will denote by $\mathcal{Q}(G,H)$ the group of all quasi-homomorphisms from $G$ to $H$ and by $\mathcal{P}(G,H)$ the subgroup of all pseudo-homomorphisms from $G$ to $H$. The group of all approximable quasi-homomorphisms from $G$ to $H$ will be denoted by $\mathcal{AQ}(G,H)$ and the group of all approximable pseudo-homomorphisms from $G$ to $H$ will be denoted by $\mathcal{AP}(G,H)$. Notice that $\mathcal{AP}(G,H) \leq \mathcal{P}(G,H)$ and $\mathcal{AQ}(G,H) \leq \mathcal{Q}(G,H)$.

**(6.3.6) Definition.** Given two topological groups $G$ and $H$, it is easily seen that the set of all the classes of algebraically splitting extensions of the form $0 \to H \to X \to G \to 0$ conforms a subgroup of $\mathrm{Ext}(G,H)$. This subgroup is denoted by $\mathrm{Ext}_0(G,H)$. One easily sees that (3.1.1.ii) remains true if we consider $\mathrm{Ext}_0$ instead of Ext. Notice that in virtue of (1.1.5) if $G$ is free or $H$ is divisible, $\mathrm{Ext}_0(G,H) = \mathrm{Ext}(G,H)$.

**(6.3.7) Theorem.** *Let $G, H$ topological abelian groups. The following maps are monomorphisms of abelian groups*

$$\begin{array}{rccc} \varphi_1 : & \mathcal{Q}(G,H)/\mathcal{AQ}(G,H) & \longrightarrow & \mathrm{Ext}_0(G,H) \\ & q & \longmapsto & [E_q : 0 \to H \stackrel{\imath_H}{\to} H \oplus_q G \stackrel{\pi_G}{\to} G \to 0] \end{array}$$

$$\begin{array}{rccc} \varphi_2 : & \mathcal{P}(G,H)/\mathcal{AP}(G,H) & \longrightarrow & \mathrm{Ext}_0(G,H) \\ & q & \longmapsto & [E_q : 0 \to H \stackrel{\imath_H}{\to} H \oplus_q G \stackrel{\pi_G}{\to} G \to 0] \end{array}$$

*Proof.* To check the additivity of $\varphi_1$ notice that if $q', q \in \mathcal{Q}(G,H)$, $\nabla_H \circ (q \times q') \circ \Delta_G = q' + q$ thus by (6.3.1.ii), $E_{q+q'} \equiv E_{\nabla_H \circ (q \times q') \circ \Delta_G} \equiv \nabla_H (E_q \times E_{q'}) \Delta_G$ which is a representative of $[E_q] + [E_{q'}]$. In view of (6.3.3), $\mathcal{AQ}(G,H)$ is the kernel of the map $\mathcal{Q}(G,H) \to \mathrm{Ext}_0(G,H);\ q \mapsto [E_q]$ hence $\varphi_1$ is one-to-one.

The same argument is valid for $\varphi_2$.

∎

**(6.3.8) Theorem.** *Let $G, H$ be topological abelian groups and let $\varphi_1$ and $\varphi_2$ be as in (6.3.7).*

*(i) If $H$ and $G$ are metrizable then $\varphi_1$ is a group isomorphism.*

*(ii) If $H$ is compact and $G$ is a zero-dimensional $k_\omega$-space then $\varphi_2$ is a group isomorphism.*

*(iii) If $H$ is complete metrizable and $G$ is complete metrizable and zero-dimensional then $\varphi_2$ is onto.*

*(iv) If $H$ is a Fréchet topological vector space and $G$ is a metrizable topological vector space then $\varphi_2$ is a linear isomorphism (considering on $\mathcal{P}(G,H)$ its natural vector space structure as a subspace of $H^G$, and on $\mathrm{Ext}_0(G,H)$ the one described in (4.1.1.ii)).*

*Proof.* In view of (6.3.7) and (6.3.2) it suffices to prove that every algebraically splitting extension of the form $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ admits a cross-section continuous at 0 if we are in the conditions of (i) or a continuous cross-section if we are in the conditions of (ii)-(iv).

The existence of such cross-section for $E$ in (i)-(iv) was proven in (5.1.6), (5.1.16), (5.1.9) and (5.2.2) respectively. Notice that in (iv) $\varphi_2$ is linear because, defining for every $\lambda \in \mathbb{R}$, $\phi_\lambda : H \to H;\ h \mapsto \lambda h$, by (6.3.1.ii) we have that $E_{\lambda q} \equiv E_{\phi_\lambda \circ q} \equiv \phi_\lambda E_q = \lambda \cdot E_q$ (see (4.1.1.iii)). ■

**(6.3.9) Corollary.** *If $Y$ is a Fréchet space and $Z$ is a metrizable topological vector space then*

$$\mathcal{P}(Z,Y)/\mathcal{AP}(Z,Y) \cong \mathrm{Ext}(Z,Y) \cong \mathrm{Ext}_{TVS}(Z,Y).$$

## §6.4 Quasi-homomorphisms to ℝ and 𝕋

**(6.4.1) Hyers' Lemma.** Although much more general than what we need, Hyers' Lemma has interesting applications to quasi-homomorphisms. The following version of this result can be found in [HIR12, Cor. 1.2].

*(i) (Hyers) Let $G$ be an abelian group (no topology assumed) and $B$ be a Banach space. Suppose $q : G \to B$ is a mapping such that*

$$\|\Delta_q(x,y)\|_Y = \|q(x+y) - q(x) - q(y)\|_B \leq \varepsilon \ \forall x, y \in G.$$

*Then there exists an additive mapping $a : G \to B$ such that $\|q(x) - a(x)\|_B \leq \varepsilon \ \forall x \in G$.*

A version of (i) for maps from an amenable group to the unit circle was obtained by Cabello in [Cab00, Cor. 1]. The following fact appears as Lemma 6 in [Cab03] and is an easy consequence of (i).

*(ii) (Cabello) Let $G$ be an abelian group (no topology assumed) and let $q : G \to \mathbb{T}$ be any mapping such that for some $\varepsilon < 1/6$*

$$\Delta_q(G \times G) = \{q(x+y) - q(x) - q(y) : x, y \in G\} \subset [-\varepsilon, \varepsilon] + \mathbb{Z}$$

*Then there exists a unique homomorphism $a : G \to \mathbb{T}$ such that $q(x) - a(x) \in [-\varepsilon, \varepsilon] + \mathbb{Z}$ for all $x \in G$.*

**(6.4.2) Notation.** For every abelian group $G$, every $V \subset G$ with $0 \in V$ and every $n < \omega$ we define $(1/2^n)V = \{x \in V : 2^k x \in V \ \forall k \in \{0, 1, \cdots, n\}\}$. Notice that if $G$ is a topological abelian group $V \in \mathcal{N}_0(G) \Rightarrow (1/2^n)V \in \mathcal{N}_0(G)$ since the homomorphism $h_k : x \in G \mapsto kx \in G$ is continuous and $(1/2^n)V = \bigcap_{k=0}^n h_{2^k}^{-1}(V)$.

**(6.4.3) Lemma.** *Let $G$ and $M$ be topological abelian groups and let $q : G \to M$ be a quasi-homomorphism.*

*(i) If $M = \mathbb{T}$ and there exists $U \in \mathcal{N}_0(G)$ such that $q(U) \subset W = [-\beta, \beta] + \mathbb{Z}$ for some $\beta < 1/6$, then $q$ is continuous at $0$.*

*(ii) If $M$ is a Banach space and $q$ maps a neighborhood of zero $U$ to a bounded subset of $M$ then $q$ is continuous at $0$.*

*Proof.* (i). Since $q$ is a quasi-homomorphism,

*($*$) for every $\rho > 0$ there exists $W_\rho \in \mathcal{N}_0(G)$ with $2q(u) - q(2u) \in [-\rho, \rho] + \mathbb{Z} \ \forall u \in W_\rho$.*

Pick any $\varepsilon > 0$. Let us find $V \in \mathcal{N}_0(G)$ with

$$q(V) \subset [-\varepsilon, \varepsilon] + \mathbb{Z}. \tag{E51}$$

Let us start taking any $\beta' \in (\beta, 1/6)$. Find $N < \omega$ with $\beta'/2^N \leq \varepsilon$. We will show that the desired neighborhood is

$$V = (1/2^N)U \cap (1/2^{N-1})W_{(\beta'-\beta)/(N2^{N-1})}.$$

The following fact is a consequence of [CD03, Cor. 2].

*($**$) Let $\rho \in (0, 1/3)$. $2^j x \in [-\rho, \rho] + \mathbb{Z} \ \forall j \leq k \Rightarrow x \in [-\rho/2^k, \rho/2^k] + \mathbb{Z}$.*

To prove that $V$ satisfies (E51), it suffices to show that

$$\forall v \in V \quad \forall n \in \{0, 1, 2, \cdots, N\}, \quad 2^n q(v) \in [-\beta', \beta'] + \mathbb{Z} \tag{E52}$$

this is because, in view of $(**)$, (E52) will imply $q(v) \in [-\beta'/2^N, \beta'/2^N]+\mathbb{Z} \subset [-\varepsilon, \varepsilon] + \mathbb{Z}$. Now, for every $n \in \{0, 1, 2, \cdots, N\}$

$$\begin{aligned}
2^n q(v) &= 2^n q(v) - 2^{n-1}q(2v) + 2^{n-1}q(2v) = 2^{n-1}\big(2q(v) - q(2v)\big) + 2^{n-1}q(2v) \\
&= 2^{n-1}\big(2q(v) - q(2v)\big) + 2^{n-1}q(2v) - 2^{n-2}q(2\cdot 2v) + 2^{n-2}q(2\cdot 2v) \\
&= 2^{n-1}\big(2q(v) - q(2v)\big) + 2^{n-2}\big(2q(2v) - q(2\cdot 2v)\big) + 2^{n-2}q(2\cdot 2v) \\
&= \cdots \\
&= \Big(\sum_{j=0}^{n-1} 2^{n-j-1}\big(2q(2^j v) - q(2\cdot 2^j v)\big)\Big) + q(2^n v). \qquad \text{(E53)}
\end{aligned}$$

Since $v \in (1/2^N)U$, we have that

$$q(2^n v) \in q(U) \in [-\beta, \beta] + \mathbb{Z}. \qquad \text{(E54)}$$

Now, for every $j \in \{0, \cdots, n-1\}$ and every $n \in \{0, \cdots, N\}$ since $2^j v \in W_{(\beta'-\beta)/(N2^{N-1})}$, by $(*)$

$$2q(2^j v) - q(2\cdot 2^j v) \in \Big[-\frac{\beta'-\beta}{N2^{N-1}}, \frac{\beta'-\beta}{N2^{N-1}}\Big] + \mathbb{Z}.$$

Which implies that

$$\begin{aligned}
2^{n-j-1}\Big(2q(2^j v) - q(2\cdot 2^j v)\Big) &\in 2^{n-j-1}\Big(\Big[-\frac{\beta'-\beta}{N2^{N-1}}, \frac{\beta'-\beta}{N2^{N-1}}\Big] + \mathbb{Z}\Big) \\
&\subset \Big[-\frac{\beta'-\beta}{N}, \frac{\beta'-\beta}{N}\Big] + \mathbb{Z}.
\end{aligned}$$

Thus,

$$\sum_{j=0}^{n-1} 2^{n-j-1}\big(2q(2^j v) - q(2\cdot 2^j v)\big) \in [-\beta'+\beta, \beta'-\beta] + \mathbb{Z}. \qquad \text{(E55)}$$

Consequently, combining (E54), (E55) and (E53), $2^n q(v) \in [-\beta', \beta'] + \mathbb{Z}$.

(ii). This is [Cab03, Lemma 5]. Nevertheless, notice that the argument used in (i) stands for $M$ Banach if we use balls of $M$ instead of neighborhoods of the form $[-\beta, \beta] + \mathbb{Z}$. Furthermore, in this case, the analogous version of $(**)$ is trivial. ■

**(6.4.4) Proposition.** *Let $M$ be either a Banach space $B$ or the unit circle $\mathbb{T}$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of topological abelian groups such that for every $\alpha < \kappa$, every quasi-homomorphism of $G_\alpha$ to $M$ is approximable. Then every quasi-homomorphism of $\prod_{\alpha<\kappa} G_\alpha$ to $M$ is approximable.*

*Proof.* We start with the case in which we have the product of just two groups i.e. assuming that $\kappa = 2$. Let $G$ and $H$ be topological abelian groups with the property specified in the proposition and fix a quasi-homomorphism $q\colon G \times H \to M$. Making use of $\Delta_q(\cdot, 0, \cdot, 0)$ we deduce that $q(\cdot, 0)$ is a quasi-homomorphism. Similarly, $q(0, \cdot)$ is a quasi-homomorphism. By hypothesis, there exist continuous homomorphisms $f_1\colon G \to M$ and $f_2\colon H \to M$ such that both $q(\cdot, 0) - f_1(\cdot)$ and $q(0, \cdot) - f_2(\cdot)$ are continuous at 0. Consider the homomorphism $f : G \times H \to M$; $(g, h) \mapsto f_1(g) + f_2(h)$. Notice that

$$\begin{aligned} q(g,h) - f(g,h) &= q(g,h) - f_1(g) - f_2(h) \\ &= \big(q(g,h) - q(g,0) - q(0,h)\big) + \big(q(g,0) - f_1(g)\big) + \big(q(0,h) - f_2(h)\big). \end{aligned} \tag{E56}$$

Using the continuity at 0 of $\Delta_q(\cdot, 0, 0, \cdot)$ we obtain that the first summand is also jointly continuous at 0 which implies that $q - f$ is continuous at 0.

The proof for a finite ordinal $\kappa$ is an easy induction. We will proceed now with the proof of the case in which $\kappa$ is an arbitrary ordinal (notice that this argument is similar to the last step in the proof of [Cab03, Th. 1]).

Let $q\colon \prod_{\alpha<\kappa} G_\alpha \to M$ be a quasi-homomorphism and let $W_0$ be the neighborhood $[-1/7, 1/7] + \mathbb{Z}$ (in the case $M = \mathbb{T}$) or the unit ball $B(0, 1)$ (in the case $M = B$). Let $W = (-1/21, 1/21) + \mathbb{Z}$ (if $M = \mathbb{T}$) or $W = B(0, 1/3)$ (if $M = B$). Using the continuity of $\Delta_q$ find an finite subset $F \subset \{\alpha < \kappa\}$ and neighborhoods $U_\alpha \in \mathcal{N}_0(G_\alpha)$, for each $\alpha \in F$, such that

$$\Delta_q\Big(\prod_{\alpha\in F} U_\alpha \times \prod_{\alpha<\kappa, \alpha\notin F} G_\alpha\Big) \subset W \tag{E57}$$

Put $J = \{\alpha < \kappa : \alpha \notin F\}$ and write $\prod_{\alpha<\kappa} G_\alpha$ as $G_1 \times G_2$, where $G_1 = \prod_{\alpha\in F} G_\alpha$ and $G_2 = \prod_{\alpha\in J} G_\alpha$. By the previous step, the quasi-homomorphism $q(\cdot, 0_{G_2})$ is approximable and there exists a homomorphism $f_1\colon G_1 \to M$ and $V_\alpha \in \mathcal{N}_0(G_\alpha)$ for each $\alpha \in F$ such that

$$\Big(q(\cdot, 0_{G_2}) - f_1(\cdot)\Big)\Big(\prod_{\alpha\in F} V_\alpha\Big) = q\Big(\prod_{\alpha\in F} V_\alpha \times \{0_{G_2}\}\Big) - f_1\Big(\prod_{\alpha\in F} V_\alpha\Big) \subset W \tag{E58}$$

We may assume that $V_\alpha \subset U_\alpha$. Further, (6.4.1) implies that there is a continuous homomorphism $f_2\colon G_2 \to M$ such that

$$\big(q(0_{G_1}, \cdot) - f_2(\cdot)\big)(G_2) = q(\{0_{G_1}\} \times G_2) - f_2(G_2) \subset W \tag{E59}$$

Using a decomposition as in (E56) and combining (E57), (E58) and (E59) we can find $O_1 \in \mathcal{N}_0(G_1)$ and $O_2 \in \mathcal{N}_0(G_1)$ such that for every $g \in O_1$ and $h \in O_2$, we can express $q(g, h) - f_1(g) - f_2(h)$ as the addition of three elements in $W$ and so $q(g, h) - f_1(g) - f_2(h)$ belongs to $W + W + W \subset W_0$. It only remains to apply (6.4.3) to deduce that the quasi-homomorphism $(g, h) \mapsto q(g, h) - f_1(g) - f_2(h)$ is continuous at 0. This completes the proof. ■

The following immediate consequence of (6.4.4) will be improved in (7.1.8).

**(6.4.5) Corollary.** *Let $M$ be either a Banach space or $\mathbb{T}$. If $\{G_n : n < \omega\}$ is a sequence of metrizable abelian groups such that $\text{Ext}(G_n, H) = 0$ for every $n < \omega$, then $\text{Ext}(\prod_{n<\omega} G_n, H) = 0$.*

*Proof.* Clearly the group $G = \prod_{n<\omega} G_n$ is metrizable. By (6.3.8), it suffices to show that every quasi-homomorphism $q : G \to H$ is approximable, which follows from (6.4.4). ■

**(6.4.6) Theorem.** *Let $M$ be either a Banach space $B$ or the unit circle $\mathbb{T}$ and let $\mu$ be a non-atomic $\sigma$-finite measure on a set $\Delta$. Let $L_0 = L_0(\mu)$ be the space of all measurable functions on $\Delta$ with the norm*

$$\|f\|_0 := \int_\Delta \min\{1,\ |f(t)|\}\, d\mu(x)$$

*Every quasi-homomorphism $q : L_0 \to M$ is approximable. Consequently* $\text{Ext}(L_0, M) = 0$.

*Proof.* We can assume, without loss of generality, that $\mu$ is a probability (note that $L_0(\mu)$ is topologically isomorphic to $L_0(\nu)$, where $\nu$ is a probability with the same null sets as $\mu$).

Let $q : L_0 \to M$ be a quasi-homomorphism, pick $\beta < 1/3$ and consider $W = [-\beta, \beta] + \mathbb{Z}$ if $M = \mathbb{T}$ or $W = B(0, \beta)$ if $M = B$. Choose $\delta_0$ such that $\Delta_q(f, g) \in W$ for every $f$, $g$ with $\|f\|_0, \|g\|_0 \leq \delta_0$.

Let $\Delta = \bigoplus_{n=1}^r \Delta_n$ be a partition of $\Delta$ into measurable sets, with $\mu(\Delta_n) \leq \delta_0$ for all $1 \leq n \leq r$. Then $L_0 = \prod_{n=1}^r L_0(\Delta_n)$ is a topological direct product. For all $f \in L_0(\Delta_n)$ we have that

$$\begin{aligned}\|f\|_0 = \int_\Delta \min\{1,\ |f(t)|\}\, d\mu(x) &\leq \int_{\{t\in\Delta : f(t)\neq 0\}} 1 d\mu(x) \\ = \mu\{t \in \Delta : f(t) \neq 0\} &\leq \mu(\Delta_n) \leq \delta_0\end{aligned}$$

Call $q_n$ the restriction of $q$ to each $L_0(\Delta_n)$. As $\Delta_{q_n}(f, g) \in W$ for every $1 \leq n \leq r$ and $f, g \in L_0(\Delta_n)$, we can apply (6.4.1) to obtain continuous homomorphisms $a_n : L_0(\Delta_n) \to M$ such that $q_n(f) - a_n(f) \in W$ for every $f \in L_0(\Delta_n)$. By (6.4.3), we have that each $q_n - a_n$ is continuous at the origin of $L_0(\Delta_n)$ and thus, the continuous homomorphism $a : L_0 \to \mathbb{T}$ given by $a(f) = \sum_{n=1}^r a_n(f_n)$ (where $f = \sum_{n=1}^r f_n,\ f_n \in L_0(\Delta_n)$) approximates $q$. ■

**(6.4.7) Protodiscrete groups.** We say that a topological abelian group $(G, \tau)$ is a *protodiscrete* group (or that the topology $\tau$ is *linear*) if it has a basis of neighborhoods of 0 formed by open subgroups. Note that protodiscrete Hausdorff groups are exactly the subgroups of products of discrete groups.

*Fact. Let $M$ be either a Banach space $B$ or the unit circle $\mathbb{T}$ and let $G$ be a protodiscrete topological abelian group. Every quasi-homomorphism $q : G \to M$ is approximable.*

*Proof.* Let $W = [-1/4, 1/4] + \mathbb{Z}$ if $M = \mathbb{T}$ or the unit ball if $M = B$. There exists an open subgroup $U \leq G$ such that $\Delta_q(U \times U) \subset W$. Using (6.4.1) we deduce that there exists an homomorphism $a : U \to M$ with $q(u) - a(u) \in W$ for every $u \in U$. Now (6.4.3) implies that any algebraic extension of $a$ (which exists because $M$ is divisible) approximates $q$. □

**(6.4.8) Notes.** The definition of $\mathcal{N}$ in (6.2.1) appears in [Cab03, Lemma 2]. The equivalence (ii)⇔(iv) in (6.3.2) was proven by Cabello in [Cab03, Lemma 10] and the argument is based on a similar result in the framework of topological vector spaces that was developed by Domański in [Dom84, Lemma 3.2]. The Fact 1 in (6.3.3) is [Cab03, Lemma 3]. (6.4.3.i) is [BCD13, Lemma 36]. The case $M = \mathbb{T}$ in (6.4.4) is [BCDT16, Lemma 1.7]. The case $M = B$ in (6.4.6) is [Cab04, Th. 2], and the case $M = \mathbb{T}$ is [BCD13, Cor. 40]. (6.4.7) was proven in [BCD13, Prop. 42].

# Chapter 7

# Extensions of topological abelian groups by $\mathbb{T}$ and $\mathbb{R}$

In this final chapter we will focus our attention towards the extensions of topological abelian groups of the form $0 \to M \to X \to G \to 0$ where $M$ is $\mathbb{R}$ or $\mathbb{T}$. We will start applying the techniques developed until this point to find properties on $G$ that force the extensions of the previous form to split.

Most of this dissertation has been focused in finding splitting extensions and it turns out that finding non-splitting ones can also be a hard problem. To conclude, in §7.2 we give several examples of non-splitting extensions of topological abelian groups of the form $0 \to M \to X \to G \to 0$ where $M$ is $\mathbb{R}$ or $\mathbb{T}$.

## §7.1 Splitting extensions by $\mathbb{R}$ and $\mathbb{T}$

**(7.1.1) $\mathbb{R}$ and $\mathbb{T}$ in $\mathcal{L}$.** The unit circle and the real numbers play a very important role in the category of locally compact abelian groups. It is known that both $\mathbb{R}$ and $\mathbb{T}$ are *injective* in $\mathcal{L}$ which means that every extension of topological abelian groups of the form $0 \to M \to X \to G \to 0$ splits if $M$ is either $\mathbb{R}$ or $\mathbb{T}$ and $G \in \mathcal{L}$. We will include a proof of this fact for the sake of completeness (for more information on this subject see [Arm81, 9.12, 9.17]):

*Fact. Let $M$ be $\mathbb{R}$ or $\mathbb{T}$ and let $G \in \mathcal{L}$. Then* $\mathrm{Ext}(G, M) = 0$.

*Proof.* Pick any extension $E : 0 \to M \xrightarrow{\imath} X \to G \to 0$. Since local compactness is a three space property $X \in \mathcal{L}$. It is well known that every group $X \in \mathcal{L}$ has the property that for every $H \leq X$ closed and every continuous homomorphism $\chi : H \to M$ there exists a continuous homomorphism $\overline{\chi} : X \to M$ such that $\overline{\chi}_{|H} = \chi$ (see [HR62, 24.12] if $M = \mathbb{T}$ and [HR62, 24.36] if $M = \mathbb{R}$). Thus the continuous homomorphism $\imath^{-1} : \imath(M) \to M$ can be extended to a continuous homomorphism $P : X \to M$. Since $P \circ \imath(m) = \imath^{-1}(\imath(m)) = m \; \forall m \in M$, by (2.1.7) $E$ splits. □

**(7.1.2) Lemma.** *Let $K$ be a compact subgroup of a topological abelian group $X$. $X^\wedge$ separates points of $K$ if and only if $K$ is dually embedded in $X$.*

*Proof.* Suppose that $X^\wedge$ separates points of $K$. It is known that for any locally compact abelian group $G$, a subgroup $L \leq G^\wedge$ is dense in $G^\wedge$ if and only if it separates points of $G$. The subgroup $L$ of $K^\wedge$ formed by all restrictions of characters of $X$ separates points of $K$ by hypothesis. Hence $L$ is dense in $K^\wedge$ and, since $K^\wedge$ is discrete, $L$ coincides with $K^\wedge$.

Suppose that $K$ is dually embedded in $X$. As $K$ is compact, it is a MAP group. Fix a nonzero $x \in K$. There exists $\chi \in K^\wedge$ such that $\chi(x) \neq 0 + \mathbb{Z}$. Since $K$ is dually embedded in $X$, there exists an extension $\widetilde{\chi} \in G^\wedge$ of $\chi$ with $\widetilde{\chi}(x) = \chi(x) \neq 0 + \mathbb{Z}$. ■

**(7.1.3) Theorem.** *Let $E : 0 \to \mathbb{T} \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. The following are equivalent:*

*(i) $E$ splits.*

*(ii) $X^\wedge$ separates points of $\imath(\mathbb{T})$.*

*(iii) $\imath(\mathbb{T})$ is dually embedded in $X$.*

*Proof.* The equivalence $(ii) \Leftrightarrow (iii)$ follows from (7.1.2).

$(iii) \Rightarrow (i)$. Suppose that $\imath(\mathbb{T})$ is dually embedded in $X$. Hence there exists a continuous character $\chi : X \to \mathbb{T}$ which extends the isomorphism $\varphi : \imath(\mathbb{T}) \to \mathbb{T}$ defined by $\varphi(\imath(t)) = t$. Since $\chi \circ \imath = \mathrm{Id}_{\mathbb{T}}$, the assertion follows from (2.1.7).

$(i) \Rightarrow (ii)$. Fix $x \in \imath(\mathbb{T})$, $x = \imath(z)$ with $z \neq 0 + \mathbb{Z}$. By (2.1.7) there exists a continuous homomorphism $P : X \to \mathbb{T}$ with $P \circ \imath = \mathrm{Id}_{\mathbb{T}}$, hence $P(\imath(z)) = \mathrm{Id}_{\mathbb{T}}(z) = z \neq 0 + \mathbb{Z}$. ■

**(7.1.4) Theorem.** *Let $E : 0 \to \mathbb{T} \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. Suppose that $G$ is locally quasi-convex. Then conditions (i), (ii), (iii) of (7.1.3) are equivalent to*

*(iv) $X$ is locally quasi-convex.*

*Proof.* $(i) \Rightarrow (iv)$. If $E$ splits, $X$ is topologically isomorphic to the product of two locally quasi-convex groups, hence it is locally quasi-convex.

$(iv) \Rightarrow (iii)$. Given an extension of topological abelian groups $0 \to \mathbb{T} \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ with $X$ and $G$ locally quasi-convex, since $X$ is locally quasi-convex, in particular it is MAP, hence its compact subgroup $\imath(H)$ is dually embedded (see (1.3.9.ii)). ■

**(7.1.5) Lemma.** *Let $G$ be a topological group and let $F$ be a compact subset of $G$ containing $0$ and having a countable base $\{U_n : n < \omega\}$ in $G$. Suppose that a sequence $\gamma = \{V_n : n < \omega\} \subset \mathcal{N}_0(G)$ satisfies $V_{n+1} + V_{n+1} \subset V_n \cap U_n$ for each $n < \omega$. Then $K = \bigcap_{n\in\omega} V_n$ is a compact subgroup of $G$, $K \subset F$ and $\gamma$ is a base for $G$ at $K$. In particular $G/K$ is metrizable.*

*Proof.* This is [AT08, Lemma 4.3.10]. ■

**(7.1.6) Lemma.** *Let $G$ be an almost-metrizable topological abelian group. Every admissible subgroup $N$ of $G$ contains an admissible, compact subgroup $K$ such that $G/K$ is metrizable.*

*Proof.* Let $N$ be an admissible subgroup of $G$ and $\{W_n : n < \omega\}$ be a sequence of open, symmetric neighborhoods of $0$ in $G$ such that $W_{n+1} + W_{n+1} + W_{n+1} \subset W_n$ for each $n$ and $\bigcap_{n\in\omega} W_n = N$. Take a compact subgroup $H$ of $G$ of countable character in $G$ and let $\{U_n : n < \omega\}$ be a basis of open neighborhoods of $H$ in $G$. Find a sequence $\{V_n : n < \omega\}$ of open symmetric neighborhoods of $0$ in $G$ such that $V_{n+1} + V_{n+1} + V_{n+1} \subset W_n \cap U_n$ and $V_n \subseteq W_n$ for every $n \in \omega$. Put $K = \bigcap_{n\in\omega} V_n$. It is clear that $K$ is admissible and $K \subset N$. By (7.1.5), $K$ is a compact subgroup of $G$ and $\{V_n : n < \omega\}$ is a base for $K$ in $G$. Hence the quotient group $G/K$ is metrizable. ■

**(7.1.7) Lemma.** *Let $\{G_\alpha : \alpha < \kappa\}$ be a family of almost-metrizable abelian groups and $G = \prod_{\alpha<\kappa} G_\alpha$. Let $\mathcal{F}$ be the family of subgroups of $G$ of the form $\prod_{\alpha\in I} N_\alpha \times \prod_{\alpha\notin I} G_\alpha$ where $I$ is a countable subset of $\{\alpha < \kappa\}$ and $N_\alpha$ is a compact admissible subgroup of $G_\alpha$ such that $G_\alpha/N_\alpha$ is metrizable. Then $\mathcal{F}$ is a cofinal family of admissible subgroups of $G$.*

*Proof.* It is easy to see that every $N \in \mathcal{F}$ is admissible. Conversely, let $N$ be an admissible subgroup of $G$. Take $\{U_n : n \in \omega\}$ the family of open symmetric neighborhoods of $0$ in $G$ that define $N$. By definition of product topology, we know that for every $U_n$, there exists a family $\{V_{n,\alpha} : \alpha < \kappa\}$ and a finite subset $J_n \subset \{\alpha < \kappa\}$ with

$$V_{n,\alpha} = \begin{cases} V_{n,\alpha} \text{ is an open symmetric neighborhood of } 0 \text{ in } G_\alpha \text{ if } \alpha \in J_n \\ V_{n,\alpha} = G_\alpha \text{ if } \alpha \notin J_n \end{cases} \tag{E60}$$

satisfying that $U_n \supset \prod_{n<\omega} V_{n,\alpha}$. This implies that

$$N = \bigcap_{n\in\omega} U_n \supset \bigcap_{n\in\omega} \prod_{\alpha<\kappa} V_{n,\alpha} = \prod_{\alpha<\kappa} \bigcap_{n\in\omega} V_{n,\alpha} \tag{E61}$$

It is clear that we may assume that $V_{n+1,\alpha} + V_{n+1,\alpha} + V_{n+1,\alpha} \subset V_{n,\alpha}$. Define $N'_\alpha = \bigcap_{n\in\omega} V_{n,\alpha}$. By construction $N'_\alpha$ is an admissible subgroup of $G_\alpha$ and by (E60), $N'_\alpha = G_\alpha$ for all but countably many $\alpha < \kappa$. Now put $N_\alpha = G_\alpha$ if $N'_\alpha = G_\alpha$, and for those $\alpha$ with $N'_\alpha \neq G_\alpha$ take (using (7.1.6)) a subgroup $N_\alpha \subset N'_\alpha$ such that $N_\alpha$ is admissible, compact and $G_\alpha/N_\alpha$ is metrizable. Then, in view of (E61), $N \supset \prod_{\alpha<\kappa} N_\alpha \in \mathcal{F}$.

■

The following result is the key part of the proof of (7.1.9).

**(7.1.8) Theorem.** *Let $\{G_\alpha : \alpha < \kappa\}$ be a family of almost-metrizable abelian groups.*

*(i) If $G_\alpha$ is a MAP group and* $\operatorname{Ext}(G_\alpha, \mathbb{T}) = 0$ *for every $\alpha < \kappa$, then* $\operatorname{Ext}(\prod_{\alpha<\kappa} G_\alpha, \mathbb{T}) = 0$.

*(ii) If* $\operatorname{Ext}(G_\alpha, \mathbb{R}) = 0$ *for every $\alpha < \kappa$, then* $\operatorname{Ext}(\prod_{\alpha<\kappa} G_\alpha, \mathbb{R}) = 0$.

*Proof.* Let $G = \prod_{\alpha<\kappa} G_\alpha$ and let $M$ be either $\mathbb{T}$ or $\mathbb{R}$. Consider the family $\mathcal{F}$ defined in (7.1.7). By (3.5.7), it suffices to prove that for each $N \in \mathcal{F}$, every extension $0 \to M \to Y \to G/N \to 0$ splits. If $N \in \mathcal{F}$, then $N = \prod_{\alpha<\kappa} N_\alpha$, where $N_\alpha$ is either a compact, admissible subgroup of $G_\alpha$ or the whole $G_\alpha$, the quotients $G_\alpha/N_\alpha$ are metrizable for every $\alpha$, and $N_\alpha \neq G_\alpha$ for at most countably many $\alpha < \kappa$. Clearly $G/N \cong \prod_{\alpha<\kappa} G_\alpha/N_\alpha$. Note that because $N_\alpha$ is compact every continuous homomorphism from $N_\alpha$ to $M$ can be continuously extended to $G_\alpha$ (indeed, if $M = \mathbb{R}$ there are not any non-trivial continuous homomorphism from $N_\alpha$ to $\mathbb{R}$ and if $M = \mathbb{T}$ it is (1.3.9.ii)). Hence by (3.5.1.ii), $\operatorname{Ext}(G_\alpha/N_\alpha, M) = 0$ for every $\alpha < \kappa$. Therefore $G/N$ is topologically isomorphic to a countable product of metrizable topological groups $G_\alpha/N_\alpha$ such that $\operatorname{Ext}(G_\alpha/N_\alpha, M) = 0$. Finally by (6.4.5), $\operatorname{Ext}(G/N, M) = 0$. ■

**(7.1.9) Theorem.** *Let $G = \prod_{\alpha<\kappa} G_\alpha$ be the product of a family of topological abelian groups such that each factor $G_\alpha$ is a dense subgroup of a MAP and Čech-complete group. Assume that* $0 = \operatorname{Ext}(G_\alpha, \mathbb{R}) = \operatorname{Ext}(G_\alpha, \mathbb{T})$ *for each $\alpha < \kappa$. If $H$ is an arbitrary product of copies of $\mathbb{R}$ and $\mathbb{T}$, then* $\operatorname{Ext}(G, H) = 0$.

*Proof.* According to (3.4.1), it suffices to show that $\operatorname{Ext}(G, M) = 0$ when $M$ is either $\mathbb{R}$ or $\mathbb{T}$. Since $G_\alpha$ is a dense subgroup of a MAP and Čech-complete group $L_\alpha$, the group $\varrho G \cong \prod_{\alpha<\kappa} \varrho G_\alpha \cong \prod_{\alpha<\kappa} L_\alpha$ is a product of Čech-complete groups (and therefore almost-metrizable groups (1.3.12.i)). By (7.1.8), we have $\operatorname{Ext}(\varrho G, M) = 0$. It now follows from (3.2.4) that $\operatorname{Ext}(G, M) = 0$. ■

**(7.1.10) Consequences on locally precompact groups.** Locally precompact groups are by definition dense subgroups of locally compact groups which are MAP and Čech-complete. By (7.1.1), both $\mathrm{Ext}(G,\mathbb{T})$ and $\mathrm{Ext}(G,\mathbb{R})$ are trivial for each $G \in \mathcal{L}$. Hence the next corollary follows from (7.1.9).

*Corollary. Let* $G = \prod_{\alpha<\kappa} G_\alpha$ *be the product of a family of locally precompact abelian groups. If* $H$ *is an arbitrary product of copies of* $\mathbb{R}$ *and* $\mathbb{T}$*, then* $\mathrm{Ext}(G,H) = 0$.

Notice that this generalizes (6.3.4.i), which is Th. 1(a) in [Cab03].

## §7.2 Examples of non-splitting extensions

**(7.2.1) Extensions of $\ell^1$.** Kalton ([Kal78]) Ribe ([Rib79]) and Roberts ([Rob77]) proved independently that there exists a non-splitting extension of topological vector spaces of the form $0 \to \mathbb{R} \to X \to \ell^1 \to 0$. A proof of this result is included here for the sake of completeness. We will need first the following two technical facts.

*Fact 1. Let* $f : X \to Y$ *be an homogeneous map between normed spaces (not necessarily a homomorphism).* $f$ *is continuous at* $0$ *if and only if it is bounded i.e. if there exists* $M \in \mathbb{R}$ *such that*

$$\|f(x)\| \;\leq\; M\|x\| \;\forall x \in X.$$

*Proof.* Suppose that $f$ is continuous at 0. There exists $\delta > 0$ such that $\|f(x)\| \leq 1 \;\forall x : \|x\| \leq \delta$. Consequently, taking $M = 1/\delta$, for every $x \in X$ we have

$$\begin{aligned}\|f(x)\| &= \left\| f\Big(\frac{\|x\|}{\delta}\cdot\frac{\delta x}{\|x\|}\Big)\right\| = \Big(\frac{\|x\|}{\delta}\Big)\left\| f\Big(\frac{\delta x}{\|x\|}\Big)\right\| \\ &\leq \frac{\|x\|}{\delta}\cdot 1 = M\|x\|.\end{aligned}$$

The converse implication is immediate. □

*Fact 2. For every* $s, t \in \mathbb{R}$*, (using the convention* $0\log 0 = 0$*)*

$$\big|s\log|s| + t\log|t| - (s+t)\log|s+t|\big| \leq |s| + |t|. \tag{E62}$$

*Proof.* Assume first that $s, t > 0$, then

$$\begin{aligned}&\big|-s\log|s| - t\log|t| + (s+t)\log|s+t|\big| \\ &\qquad= \big|s\big(\log(s+t) - \log s\big) + t\big(\log(s+t) - \log t\big)\big| \\ &\qquad= \Big|-s\log\big(\frac{s}{s+t}\big) - t\log\big(\frac{t}{s+t}\big)\Big| = -s\log\big(\frac{s}{s+t}\big) - t\log\big(\frac{t}{s+t}\big)\Big| \\ &\qquad= (s+t)\Big(-\frac{s}{s+t}\log\big(\frac{s}{s+t}\big) - \frac{t}{s+t}\log\big(\frac{t}{s+t}\big)\Big) \qquad\qquad (E63) \\ &\qquad\leq (s+t)(2/e) \leq s+t. \qquad\qquad (E64)\end{aligned}$$

In (E63) we have used that if $0 \leq x \leq 1$, $|x \log x| \leq 1/e$. If $s, t < 0$, apply (E64) to $-s$ and $-t$.

It suffices to show (E62) for the case in which $s+t > 0$, $t > 0$ and $s < 0$. If we apply the known case of (E62) replacing $s$ by $-s$ and $t$ by $s+t$ we obtain

$$\big|t \log |t| + s \log |s| - (s+t) \log |s+t|\big| \leq -s + s + t \leq |s| + |t|,$$

which completes the proof. □

*(i)* $\mathrm{Ext}(\ell^1, \mathbb{R}) \neq 0$.

*Proof.* Consider the following dense subspace of $\ell^1$:

$$D = \big\{x = \{x_n : n < \omega\} \in \ell^1 : \exists N < \omega \text{ such that } x_n = 0 \; \forall n \geq N\big\}.$$

By (3.2.5), since $\mathbb{R}$ is Čech-complete, it suffices to show that $\mathrm{Ext}(D, \mathbb{R}) \neq 0$. According to (6.3.8.i) it is sufficient to find a non-approximable quasi-homomorphism from $D$ to $\mathbb{R}$. Define for every $x \in D$

$$q(x) = \sum_{n\in\omega} \Big(x_n \log |x_n|\Big) - \Big(\sum_{n\in\omega} x_n\Big) \log \Big|\Big(\sum_{n\in\omega} x_n\Big)\Big| \tag{E65}$$

and let us see that $q : D \to \mathbb{R}$ is a non-approximable quasi-homomorphism. Pick $x = \{x_n\}, y = \{y_n\} \in D$. Using (E62) (with $s = x_n$, $t = y_n$)

$$\big|(x_n + y_n) \log |x_n + y_n| - x_n \log |x_n| - y_n \log |y_n|\big| \leq |x_n| + |y_n|. \tag{E66}$$

Notice that if two sequences $\{a_n\}, \{b_n\} \in D$ satisfy that $|a_n| \leq |b_n|$, then $|\sum_{n\in\omega} a_n| \leq \sum_{n\in\omega} |b_n|$. Having this in mind, summing over $n$ in (E66) we obtain

$$\left|\sum_{n\in\omega}(x_n + y_n) \log |x_n + y_n| - \sum_{n\in\omega} x_n \log |x_n| - \sum_{n\in\omega} y_n \log |y_n|\right| \leq \|x\| + \|y\|. \tag{E67}$$

On the other hand, invoking again (E62) (this time with $s = \sum_{n\in\omega} x_n$, $t = \sum_{n\in\omega} y_n$)

$$\begin{aligned} &\Bigg|\Big(\sum_{n\in\omega}(x_n + y_n)\Big) \log \Big|\sum_{n\in\omega}(x_n + y_n)\Big| \\ &\quad - \Big(\sum_{n\in\omega} x_n\Big) \log \Big|\sum_{n\in\omega} x_n\Big| - \Big(\sum_{n\in\omega} y_n\Big) \log \Big|\sum_{n\in\omega} y_n\Big|\Bigg| \leq \|x\| + \|y\|. \end{aligned} \tag{E68}$$

Using (E67) and (E68) we obtain that $|q(x+y) - q(x) - q(y)| \leq 2(\|x\| + \|y\|)$, which implies that $\Delta_q$ is continuous and consequently $q$ is a quasi-homomorphism.

Suppose now that $q$ is approximable. Then there exists a homomorphism $a : D \to \mathbb{R}$ such that $q - a$ is continuous at 0. Notice that it follows easily from (E65) that $q$ is homogeneous. Consider for each $m \in \omega$ the canonical vector $e^m \in D$ such that $e^m_n = 0$ if $n \neq m$ and $e^m_m = 1$. Since $q - a$ is continuous at 0 and $q(\lambda e^m) = \lambda q(e^m) = 0\ \forall \lambda \in \mathbb{R}$, the restriction $a$ to the line generated by $e^m$ is a continuous homomorphism. Every continuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ is homogeneous, therefore the restriction of $a$ to the line $\{\lambda e^m : \lambda \in \mathbb{R}\}$ is homogeneous. Since the vectors $e^m$ generate $D$, $a$ is homogeneous. We have proven that the map $q - a$ is homogeneous and therefore it is bounded (see Fact 1). Thus there exists $M \in \mathbb{R}$ with

$$|q(x) - a(x)| \leq M\|x\|,\ \forall x \in D. \tag{E69}$$

Using that $q(e^m) = 0$, $|a(e^m)| \leq M$ for all $m < \omega$, consequenly $|a(x)| \leq M\|x\|\ \forall x \in D$ and by (E69)

$$|q(x)| \leq 2M\|x\|,\ \forall x \in D. \tag{E70}$$

However

$$q(1/m(e^1 + \cdots + e^m)) = -(1/m)m \log m = -\log m,$$

which contradicts (E70). □

*(ii)* $\operatorname{Ext}_{TVS}(\ell^1, \mathbb{R}) \neq 0$.

*Proof.* Since $\ell^1$ is a Banach space, in particular it is locally bounded, then by (4.2.4) $\operatorname{Ext}_{\mathrm{TVS}}(\ell^1, \mathbb{R}) \cong \operatorname{Ext}(\ell^1, \mathbb{R})$, which according to (i) is non-trivial. □

**(7.2.2) Theorem.** *Let $Y$ be a topological vector space and let $\rho : \mathbb{R} \to \mathbb{T}$ be the canonical projection. The map $\varphi : \operatorname{Ext}_{TVS}(Y, \mathbb{R}) \to \operatorname{Ext}(Y, \mathbb{T})$ defined by $\varphi([E]) = [\rho E]$ is a monomorphism.*

*Proof.* By (3.1.1.ii) $\varphi$ is a homomorphism of abelian groups. To see that it is one-to-one, pick an extension of topological vector spaces $E : 0 \to \mathbb{R} \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ and assume that $\rho E$ splits. Using (2.2.2) we obtain the following commutative diagram

$$\begin{array}{lccccccccc}
E: & 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\imath} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle s} & & \| & & \\
\rho E: & 0 & \longrightarrow & \mathbb{T} & \xrightarrow[r]{} & PO & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

As $\rho E$ splits there exists a continuous homomorphism $P : PO \to \mathbb{T}$ for such that $P \circ r = \mathrm{Id}_{\mathbb{T}}$ and consequently $P \circ s \circ \imath = \rho$. Since $X$ is a topological vector space, $P \circ s$ is of the form $x \mapsto \rho(f(x)) = f(x) + \mathbb{Z}$ for some continuous linear mapping $f : X \to \mathbb{R}$ (see [HR62, 23.32]). This clearly implies $f \circ \imath = \mathrm{id}_{\mathbb{R}}$, hence $E$ splits. ■

**(7.2.3) Corollary.** $\mathrm{Ext}(\ell^1, \mathbb{T}) \neq 0$.

*Proof.* By (7.2.2) $\mathrm{Ext}(\ell^1, \mathbb{T})$ contains an isomorphic copy of $\mathrm{Ext}_{\mathrm{TVS}}(\ell^1, \mathbb{R})$ which is non-trivial in view of (7.2.1.ii). ■

**(7.2.4) Stevens' group topologies on $\mathbb{R}^n$.** Let $\{v_j\}$ be a sequence in $\mathbb{R}^n$ and $\{p_j\}$ be a sequence in $\mathbb{R}$. $\|\cdot\|$ will represent the Euclidean norm. The pair $(\{v_j\}, \{p_j\})$ is called a *sequential norming pair* (shortly SNP) if it satisfies the following properties:

*(a)* $0 < p_{j+1} \leq p_j \ \forall j < \omega$ *and* $\lim_{j\to\infty} p_j = 0$.

*(b)* $0 < \|v_j\| \leq \|v_{j+1}\|$.

*(c)* $\inf\{p_{j+1}\|v_{j+1}\|/\|v_j\| : j < \omega\} > 0$.

The following result provides a way to define group topologies on $\mathbb{R}^n$ using sequential norming pairs:

*(i) Let $(\{v_j\}, \{p_j\})$ be a SNP. The map*

$$\begin{array}{rccc} \nu: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & \inf\left\{ \sum_{j<\omega} |c_j| p_j + \left\| x - \sum_{j<\omega} c_j v_j \right\| : \{c_j\} \in \mathbb{Z}^{(\omega)} \right\} \end{array} \qquad \text{(E71)}$$

*is a group-norm on $\mathbb{R}^n$ such that $\nu(x) \leq \|x\| \ \forall x \in \mathbb{R}^n$ and $\nu(v_j) \leq p_j \ \forall j < \omega$. $\nu$ induces a metrizable group topology $\tau_\nu$ on $\mathbb{R}^n$, weaker that the usual, in which $\lim_{j\to\infty} v_j = 0$ ([Ste82, Proposition 4.1]).*

If $\nu$ is the group-norm associated to the SNP $(\{v_j\}, \{p_j\})$, we say that $(\{v_j\}, \{p_j\}, \nu)$ is a *sequential norming triple* (SNT) for $\mathbb{R}^n$.

*(ii) Let $(\{v_j\}, \{p_j\}, \nu)$ SNT for $\mathbb{R}^n$, and let $\{y_j\}$ any sequence in $\mathbb{R}^m$. The map*

$$\begin{array}{rccc} \mu: & \mathbb{R}^{n+m} & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & \inf\left\{ \sum_{j<\omega} |c_j| p_j + \left\| x - \sum_{j<\omega} c_j (v_j, y_j) \right\| : \{c_j\} \in \mathbb{Z}^{(\omega)} \right\} \end{array}$$

*satisfies that $\mu(x) \leq \|x\|$ and induces a metrizable group topology $\tau_\mu$ on $\mathbb{R}^{n+m}$ in which $\lim_{j\in\omega}(v_j, y_j) = 0$. Furthermore the subgroup $\{0\}^n \times \mathbb{R}^m \leq (\mathbb{R}^{n+m}, \tau_\mu)$ is closed and inherits the usual topology (see [SS04, Prop. 5 and 7]).*

Suppose that $(\{v_j\}, \{p_j\})$ is a SNP for $\mathbb{R}^n$ and $\{y_j\}$ is any sequence in $\mathbb{R}^m$. We will call $(\{(v_j, y_j)\}, \{p_j\})$ an *extended norming pair* (shortly ENP). $(\{(v_j, y_j)\}, \{p_j\}, \mu)$ will be called an *extended norming triple* and denoted ENT.

**(7.2.5) Lemma.** *Let* $(\{v_j\},\{p_j\},\nu)$ *be a SNT for* $\mathbb{R}$ *and let* $(\{(v_j,y_j)\},\{p_j\},\mu)$ *be an ENT for* $\mathbb{R}^2$*. Consider* $\tau_\nu$*,* $\tau_\mu$ *the group topologies induced by* $\nu$*,* $\mu$ *and the maps*

$$\begin{array}{rccc} \pi: & (\mathbb{R}^2,\tau_\mu) & \longrightarrow & (\mathbb{R},\tau_\nu) \\ & (x,y) & \longmapsto & x \end{array} \qquad \begin{array}{rccc} \imath: & \mathbb{R} & \longrightarrow & (\mathbb{R}^2,\tau_\mu) \\ & y & \longmapsto & (0,y) \end{array}$$

*Then the short exact sequence* $E_{\mu\nu}: 0 \to \mathbb{R} \xrightarrow{\imath} (\mathbb{R}^2,\tau_\mu) \xrightarrow{\pi} (\mathbb{R},\tau_\nu) \to 0$ *is an extension of topological abelian groups.*

*Proof.* (7.2.4.ii) tells us that $\imath$ is an embedding. $\pi$ is continuous because $\nu(\pi(x)) = \nu(x) \leq \mu(x,y)$. Let $B_\mu(0,\delta), B_\nu(0,\delta)$ be balls of radius $\delta$ centered at zero for $\mu$ and $\nu$ respectively. Suppose that $x \in B_\nu(0;\delta)$. In view of (E71), $x$ can be written as $x = \sum c_j v_j + z$, where $\sum p_j|c_j| + |z| < \delta$. Notice that $x = \pi(\sum c_j(v_j,y_j) + (z,0))$ and

$$\begin{aligned} \mu\Big(\sum c_j(v_j,y_j) + (z,0)\Big) &\leq \mu\Big(\sum c_j(v_j,y_j)\Big) + \mu(z,0) \\ &\leq \sum |c_j|p_j + \|(z,0)\| < \delta \end{aligned}$$

therefore $x \in \pi(B_\mu(0,\delta))$ and $\pi$ is open. ∎

**(7.2.6) Proposition.** *Let* $(\{v_j\},\{p_j\},\nu)$ *be a SNT for* $\mathbb{R}$*. Choose a sequence* $\{y_j\}$ *in* $\mathbb{R}$ *such that it does not converges to* $0$ *in the usual topology and* $\lim_{j\in\omega} y_j/v_j = 0$ *in the usual topology [For instance take* $v_j = j!, y_j = j$ *and* $p_j = 1/j$*]. Take the ENT* $(\{(v_j,y_j)\},\{p_j\},\mu)$ *in* $\mathbb{R}^2$ *and* $\tau_\mu$*,* $\tau_\nu$ *the group topologies induced by* $\mu$ *and* $\nu$*.*

*(i) There is not any non-trivial continuous homomorphism form* $(\mathbb{R}^2,\tau_\mu)$ *to* $\mathbb{R}$*.*

*(ii) The extension* $E_{\mu\nu}: 0 \to \mathbb{R} \xrightarrow{\imath} (\mathbb{R}^2,\tau_\mu) \xrightarrow{\pi} (\mathbb{R},\tau_\nu) \to 0$ *constructed as in (7.2.5) does not split.*

*Proof.* (i) Let $f : (\mathbb{R}^2,\tau_\mu) \to \mathbb{R}$ be a non-trivial continuous homomorphism. $f$ must also be continuous as a map from $\mathbb{R}^2$ to $\mathbb{R}$ with the usual topologies because the topology $\tau_\mu$ is weaker than the usual. Then there exist $a, b \in \mathbb{R}$ such that $f(x,y) = ax + by$. By definition of norming triple, the sequence $\{(v_j,y_j)\}$ converges to $0$ in $(\mathbb{R}^2,\tau_\mu)$ and using the continuity of $f$ we deduce that the sequence $\{av_j + by_j\}$ converges to $0$ in $\mathbb{R}$ with the usual topology. Since $\{v_j\}$ is increasing and non-zero, the sequence $\{a + by_j/v_j\}$ also converges to $0$ in $\mathbb{R}$ and therefore $a = 0$. This implies that the sequence $\{f(v_j,y_j)\} = \{by_j\}$ must converge to $0$ in the usual topology which contradicts the assumption.

(ii) This is a trivial consequence of (i) and (2.1.7). ∎

**(7.2.7) Definition.** If $k$ is a real number, $\lfloor k \rceil$ is the greatest integer less than or equal to $k-1$.

**(7.2.8) Lemma.** *Let $\{v_j\}$ be a sequence in $\mathbb{R}^n$ such that $0 < \|v_j\| < \|v_{j+1}\|$ and $\lim_{j\to\infty} \|v_j\| = \infty$. Let $\{q_j\}$ be a sequence in $\mathbb{R}^m$. Create $\{w_j\}$ a sequence in $\mathbb{R}^{n+m}$ by defining $w_j = (v_j, q_j)$. Let $k < \omega$, $a_j, b_j$ and $c_j$ be integers, and suppose that*

$$\max\{|a_j|, |b_j|, |c_j|\} < \left\lfloor \frac{\|v_{j+1}\|}{\|v_j\|} \right\rceil \cdot \frac{1}{3}$$

*for $1 \leq j \leq k$ and that*

$$\sum_{j=1}^{k} a_j w_j + \sum_{j=1}^{k} b_j w_j = \sum_{j=1}^{k} c_j w_j.$$

*Then $a_j + b_j = c_j$ for every $1 \leq j \leq k$.*

*Proof.* This is [SS04, Lemma 10]. ■

**(7.2.9) Lemma.** *Let $(\{v_j\}, \{p_j\}, \nu)$ be a SNT for $\mathbb{R}$. Suppose that $\{v_j\}$ generates a discrete subgroup in $\mathbb{R}$ and that $p_j \lfloor |v_{j+1}|/|v_j| \rceil \geq 1\ \forall j < \omega$.*

*(i) There exists $\eta > 0$ such that for every $x$ with $\nu(x) < \eta$, there exists a unique $z \in \mathbb{R}$ and a unique sequence $\{c_j\} \in \mathbb{Z}^{(\omega)}$ such that $x = \sum c_j v_j + z$ and $\nu(x) = \sum |c_j| p_j + |z|$.*

*(ii) Let $(\{(v_j, y_j)\}, \{p_j\}, \mu)$ be an ENT for $\mathbb{R}^2$ and let $\pi : \mathbb{R}^2 \to \mathbb{R}$; $(x, y) \mapsto x$. The map $s : (\mathbb{R}, \tau_\nu) \to (\mathbb{R}^2, \tau_\mu)$ defined as*

$$s(x) = \begin{cases} \sum c_j (v_j, y_j) + (z, 0) & \text{if } \nu(x) < \eta \\ (x, 0) & \text{if } \nu(x) \geq \eta \end{cases}$$

*(where $\eta$, $\{c_j\}$ and $z$ are as in (i)) is a cross-section for $\pi$ continuous at $0$.*

*Proof.* (i). Suppose that the discrete subgroup of $\mathbb{R}$ generated by $\{v_j\}$ is $r\mathbb{Z}$ where $r > 0$. Define $\eta = \min\{r/3, 1/3\}$. Since $\nu(x) < \eta$, in view of (E71) we can always write $x = \sum c_j v_j + z$ with $\sum |c_j| p_j + |z| < \eta$. We need to show that $\nu(x) = \sum |c_j| p_j + |z|$ and that $x = \sum c_j v_j + z$ is the unique decomposition which attains the norm. Both statement are consequences of the following claim:

($*$) *If $z' \in \mathbb{R}$, $\{c'_j\} \in \mathbb{Z}^{(\omega)}$ are such that $x = \sum c'_j v_j + z'$ and $\sum |c'_j| p_j + |z'| \leq \sum |c_j| p_j + |z|$, then $z = z'$ and $c_j = c'_j\ \forall j < \omega$.*

From $\sum c_j v_j + z = \sum c'_j v_j + z'$ it follows that $z - z' \in r\mathbb{Z}$. Since $|z - z'| \leq |z| + |z'| \leq 2\eta < r$ we deduce that $z = z'$.

Since $\sum |c_j|p_j < \eta$, $|c_j|p_j < \eta \leq 1/3\ \forall j < \omega$. Using that $p_j \lfloor |v_{j+1}|/|v_j| \rceil \geq 1$, we obtain

$$|c_j| < \frac{1}{3} \cdot \frac{1}{p_j} \leq \frac{1}{3} \lfloor |v_{j+1}|/|v_j| \rceil\ \forall j < \omega.$$

Analogously $|c'_j| \leq \lfloor |v_{j+1}|/|v_j| \rceil /3\ \forall j < \omega$. Applying (7.2.8) (taking $q_j = 0\ \forall j < \omega$), we obtain $c_j = c'_j, \forall j < \omega$. □

(ii). To check that $s$ is continuous at 0, fix $\epsilon > 0$, and take $\delta < \min\{\eta, \epsilon\}$. Suppose that $x \in B_\nu(0;\delta)$. In view of (E71), $x$ can be written as $x = \sum c_j v_j + z$, where $\sum p_j |c_j| + |z| < \delta$. Since $\delta < \eta$, by ($*$) the previous decomposition of $x$ coincides with the one obtained in (i). Hence

$$\begin{aligned}\mu(s(x)) &= \mu\Big(\sum c_j(v_j, y_j) + (z,0)\Big) \leq \mu\Big(\sum c_j(v_j,y_j)\Big) + \mu(z,0) \\ &\leq \sum |c_j|p_j + \|(z,0)\| < \delta < \epsilon.\end{aligned}$$

Let us see that $\pi \circ s = \mathrm{Id}_{\mathbb{R}}$. If $\nu(x) \geq \eta$, $\pi(s(x)) = \pi(x,0) = x$. If $\nu(x) < \eta$ write $x = c_j v_j + z$, and $\pi \circ s(x) = \pi(\sum c_j(v_j,y_j) + (z,0)) = \sum c_j v_j + z = x$. ■

**(7.2.10) Theorem.** *Let $(\{v_j\},\{p_j\},\mu)$ be in the conditions of (7.2.9). Let $\{y_j\}$ be a sequence in $\mathbb{R}$ that does not converge to 0 in the usual topology and such that $\{y_j/v_j\}$ converges to 0 in the usual topology [For instance take $v_j = j!, y_j = j$ and $p_j = 1/j$].*

*Consider $\eta > 0$ as in (7.2.9.i) and the unique decomposition $x = \sum c_j v_j + z$ for every $x \in \mathbb{R}$ with $\nu(x) = \sum |c_j|p_j + |z|$. Define*

$$\begin{array}{rccc} q: & (\mathbb{R},\tau_\nu) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \left\{\begin{array}{ll} \sum c_j y_j & \text{if } \nu(x) < \eta \\ 0 & \text{if } \nu(x) \geq \eta \end{array}\right. \end{array}$$

*Then $q : (\mathbb{R},\tau_\nu) \to \mathbb{R}$ is a non-approximable quasi-homomorphism.*

*Proof.* Consider the ENT $(\{(x_j,y_j)\},\{p_j\},\mu)$. Since we are in the conditions of (7.2.6) the extension $E_{\mu\nu} : 0 \to \mathbb{R} \xrightarrow{\imath} (\mathbb{R}^2,\tau_\mu) \xrightarrow{\pi} (\mathbb{R},\tau_\nu) \to 0$ does not split.

Define $S : (\mathbb{R}^2,\tau_\mu) \to \mathbb{R}^2;\ (x,y) \mapsto (y,x)$ and $\pi_1 : \mathbb{R}^2 \to \mathbb{R};\ (x,y) \mapsto x$. Consider $s : (\mathbb{R},\nu) \to (\mathbb{R}^2,\tau_\mu)$ as in (7.2.9.ii). Pick $x \in \mathbb{R}$, if $\nu(x) < \eta$

$$\begin{aligned}\pi_1(S(s(x))) &= \pi_1\Big(S\Big(s\Big(\sum c_j v_j + z\Big)\Big)\Big) = \pi_1\Big(S\Big(\sum c_j(v_j,y_j) + (z,0)\Big)\Big) \\ &= \pi_1\Big(\sum c_j(y_j,v_j) + (0,z)\Big) = \sum c_j y_j + 0 = q(x).\end{aligned}$$

If $\nu(x) \geq \eta$

$$\pi_1(S(s(x))) = \pi_1(S(x,0)) = \pi_1(0,x) = 0 = q(x).$$

Thus $q = \pi_1 \circ S \circ s$.

Notice that $S$ witnesses the algebraic equivalence of $E$ and the trivial extension $E_0 : \mathbb{R} \to \mathbb{R} \times (\mathbb{R}, \tau_\nu) \to (\mathbb{R}, \tau_\nu) \to 0$. Accordingly, in view of the implication (ii)⇒(iv) of (6.3.2), $E_{\mu\nu}$ is equivalent to the extension $E_q : 0 \to \mathbb{R} \to \mathbb{R} \oplus_q (\mathbb{R}, \tau_\nu) \to (\mathbb{R}, \tau_\nu) \to 0$ given by the quasi-homomorphism $q = \pi_1 \circ S \circ s$. Finally, by (6.3.3), $q$ is not approximable. ∎

**(7.2.11) Theorem.** *Let $(\{v_j\}, \{p_j\}, \nu)$ be a SNT in the conditions of (7.2.9) and let $\tau_\nu$ be the group topology in $\mathbb{R}$ generated by the group-norm $\nu$. Then* $\mathrm{Ext}(\mathbb{R}, (\mathbb{R}, \nu))$ *is an infinite dimensional vector space.*

*Proof.* Notice first that in virtue of (4.1.1.iii) $\mathrm{Ext}(\mathbb{R}, (\mathbb{R}, \nu))$ is a vector space. Let $\{y_j\}$ be any increasing sequence satisfying that $\lim_{j\in\omega} y_j/v_j = 0$ in the usual topology of $\mathbb{R}$ (take for instance $y_j = \sqrt{|v_j|}$). Consider $\mathbb{P} = \{r_n : n < \omega\}$ an enumeration of the set of prime numbers and define

$$z_j^{(n)} = \begin{cases} y_j & \text{if } j = (r_n)^k \text{ for some } k < \omega \\ 0 & \text{otherwise} \end{cases} \tag{E72}$$

For every $n < \omega$, the extending norming triple $(\{(v_j, z_j^{(n)})\}, \{p_j\}, \mu_n)$ satisfies the hypothesis of (7.2.10).

Construct for every $n < \omega$ the extension $E_{\mu_n\nu} : 0 \to \mathbb{R} \to (\mathbb{R}^2, \tau_{\mu_n}) \to (\mathbb{R}, \nu) \to 0$ as in (7.2.5). Let us check that $\mathcal{E} = \{[E_{\mu_n\nu}] : n < \omega\}$ is linearly independent in $\mathrm{Ext}(\mathbb{R}, (\mathbb{R}, \nu))$ with respect to the vector space structure defined in (4.1.1.ii).

Fix $n < \omega$. Use (7.2.9) to obtain $\eta > 0$ and the unique decomposition $x = \sum c_j v_j + z$ for every $x \in \mathbb{R}$ where $\nu(x) = \sum |c_j| p_j + |z|$. Define the map $q_n : (\mathbb{R}, \nu) \to \mathbb{R}$ as

$$q_n(x) = \begin{cases} \sum c_j z_j^{(n)} & \text{if } \nu(x) < \eta \\ 0 & \text{if } \nu(x) \geq \eta \end{cases}$$

According to (7.2.10), $q_n$ is a non-approximable quasi-homomorphism. By the proof of (7.2.10) $E_{\mu_n\nu}$ is equivalent to an extension of the form $E_{q_n}$ : $0 \to H \overset{\imath_H}{\to} H \oplus_{q_n} G \overset{\pi_G}{\to} G \to 0$.

Suppose that there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}\backslash\{0\}$ such that $\lambda_1 E_{\mu_{n_1}\nu} + \cdots + \lambda_k E_{\mu_{n_k}\nu} \equiv E_0$. By (6.3.8.iv)

$$E_0 \equiv \lambda_1 E_{\mu_{n_1}\nu} + \cdots + \lambda_k E_{\mu_{n_k}\nu} \equiv \lambda_1 E_{q_{n_1}} + \cdots + \lambda_k E_{q_{n_1}} \equiv E_{(\lambda_1 q_{n_1} + \cdots \lambda_k q_{n_k})}$$

hence the quasi-homomorphism $q = \lambda_1 q_{n_1} + \cdots + \lambda_k q_{n_k}$ is approximable.

Consider the sequence $\{t_j = \lambda_1 z_j^{(n_1)} + \cdots + \lambda_k z_j^{(n_k)}\}$. Since the supports of the sequences $\{z_j^{(n_l)}\}$ are disjoint for all $l \leq k$, and they do not converge to 0, $\{t_j\}$ does not converge to 0 in the usual topology of $\mathbb{R}$. In view of (E72) the quotients $t_j/v_j$ are either $y_j/v_j$ or 0, therefore $\lim_{j\to\infty} t_j/v_j = 0$ in the usual topology. Then the ENT $(\{(v_j, t_j)\}, \{p_j\}, \mu_t)$ satisfies the hypothesis of (7.2.10). An easy verification shows that $q$ is the non-approximable quasi-homomorphism that we obtain when we apply the construction of (7.2.10) to the triple $(\{(v_j, t_j)\}, \{p_j\}, \mu_t)$, which gives us a contradiction. ■

**(7.2.12) Notes.** (7.1.2), (7.1.3) and (7.1.4) are Lemma 3, Th. 5 and Th. 7 of [BCD13] respectively. (7.1.7) is [BCDT16, Lemma 3.7] and (7.1.8.i) is [BCDT16, Prop. 1.8]. (7.1.9) is [BCDT16, Th. 3.13]. Results (7.2.5), (7.2.6), (7.2.9), (7.2.10) and (7.2.11) are part of an unpublished joint work with M. J. Chasco, X. E. Domínguez and C. Stevens.

# Index of symbols


| | |
|---|---|
| $\overline{A}$ | [Adherence of a set $A$] |
| $A(X)$ | [Free topological abelian group] (1.3.14) |
| $\mathrm{CHom}(G,H)$ | [Group of continuous homomorphisms of the form $G \to H$] |
| $\Delta_G, \nabla_H$ | (1.1.5) |
| $E$ | [Extension of topological abelian groups] |
| $f \triangle g$ | [Diagonal product of $f$ and $g$, $x \mapsto (f(x), g(x))$] (1.2.4.ii) |
| $\triangle_{\alpha<\kappa} f_\alpha$ | [Diagonal product of the family $\{f_\alpha\}$] (1.2.4.ii) |
| $E_0$ | [Trivial extension $0 \to H \to H \times G \to G \to 0$] (2.1.1) |
| $E_\tau$ | [Canonical extension $0 \to H \overset{\imath_\tau}{\to} (H \times G, \tau) \overset{\pi_\tau}{\to} G \to 0$] (2.1.5) |
| $tE$ | [Push-out extension of $E$ and $t$] (2.2.2) |
| $Et$ | [Pull-back extension of $E$ and $t$ ] (2.2.6) |
| $E_1 \times E_2$ | [Cartesian product of extensions] 23 |
| $E \equiv E'$ | [Equivalence of extensions] (2.1.1) |
| $\exp(M)$ | [Set of all closed non-empty subsets of $M$] (5.1.7) |
| $\mathrm{Ext}(G,H)$ | [Group of all extensions of topological abelian groups of $G$ by $H$] (3.1.1) |
| $\mathrm{Ext}_0(G,H)$ | [Group of all algebraically splitting extensions of topological abelian groups of $G$ by $H$] (6.3.6) |
| $\mathrm{Ext}_{\mathrm{TVS}}(Z,Y)$ | [Group of all extensions of topological vector spaces of $Z$ by $Y$] 45 |
| $\mathrm{Id}_G$ | [Identity map on $G$] |
| $\imath_H : H \to H \times G,$ | [Canonical inclusion $h \mapsto (h,0)$] 33, (2.1.5) |
| $\imath_\tau : H \to (H \times G, \tau),$ | [Canonical inclusion $h \mapsto (h,0)$] 34 |
| $\lfloor k \rceil$ | (7.2.7) |
| $\lim_{\leftarrow} \mathcal{P}$ | [Inverse limit of the system $\mathcal{P}$] (1.2.5) |
| $L_0(\mu)$ | (6.4.6) |
| $\ell^p$ | (1.4.3) |
| $\mathcal{N}_0(G)$ | [Set of all neighborhoods of the neutral element of $G$] |
| $\omega$ | [Set of natural numbers, first infinite cardinal] |
| $(PO, r, s)$ | [Push-out triple] (2.2.1) |
| $(PB, r, s)$ | [Pull-back triple] (2.2.5) |
| $\pi_G : H \times G \to G$ | [Canonical projection $(h,g) \mapsto g$] (2.1.1), (2.1.5) |
| $\pi_\tau : (H \times G, \tau) \to G$ | [Canonical projection $(h,g) \mapsto g$] (2.1.5) |
| $\mathcal{P}(G,H)$ | [Group of all pseudo-homomorphisms from $G$ to $H$] (6.3.5) |
| $\mathcal{Q}(G,H)$ | [Group of all quasi-homomorphisms from $G$ to $H$] (6.3.5) |

| | |
|---|---|
| $\mathcal{AP}(G,H)$ | [Group of all aproximable pseudo-homomorphisms from $G$ to $H$] (6.3.5) |
| $\mathcal{AQ}(G,H)$ | [Group of all aproximable quasi-homomorphisms from $G$ to $H$] (6.3.5) |
| $\mathbb{Q}$ | [Rational numbers] |
| $\mathbb{Q}_p$ | [p-adic numbers] (1.3.7) |
| $\mathbb{R}$ | [Real numbers] |
| $\rho : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ | [Canonical projection $r \mapsto r + \mathbb{Z}$] (2.1.6) |
| $\varrho G$ | [Raĭkov completion of the topological abelian group $G$] (3.2.1) |
| $\mathbb{T}$ | [Unit circle $\mathbb{R}/\mathbb{Z}$] |
| $\tau_\nu$ | [Group-topology induced by the group-norm $\nu$] (1.3.2) |
| $\mathrm{Hom}(G,H)$ | [Homomorphisms (not necessarily continuous) from $G$ to $H$] |
| $W(V,U)$ | (6.2.1) |
| $\mathbb{Z}$ | [Integer numbers] |
| $\mathbb{Z}_p$ | [$p$-adic integers] (1.3.7) |
| $0$ | [Neutral element of an abelian group] |
| $0 + H$ | [Neutral element of the quotient space $G/H$] |
| $0_G : G \to G$ | [Zero map on $G$] |

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