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A Non-linear Approach with Long Range Dependence based on Chebyshev Polynomials

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ABSTRACT

This paper examines the interaction between non-linear deterministic trends and long run dependence by means of employing Chebyshev time polynomials and assuming that the detrended series displays long memory with the pole or singularity in the spectrum occurring at one or more possibly non-zero frequencies. The combination of the non-linear structure with the long memory framework produces a model which is linear in parameters and therefore it permits the estimation of the deterministic terms by standard OLS-GLS methods. Moreover, we present a procedure that permits us to test (possibly fractional) orders of integration at various frequencies in the presence of the Chebyshev trends with no effect on the standard limit distribution of the method. Several Monte Carlo experiments are conducted and the results indicate that the method performs well, and an empirical application, using data of real exchange rates is also carried out at the end of the article.

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1. Introduction

This paper deals with the analysis of long range dependence in the context of non-linear models. In particular, we employ the Chebyshev polynomials in time to describe the deterministic part of the model, and suppose that the detrended series displays long memory behavior. We use a general definition of long memory that allows the inclusion of one or more poles or singularities in the spectrum at various frequencies. Thus, we consider the standard case of $I(d, d > 0)$ behavior, but also other possibilities such as seasonal/cyclical long range dependence and multiple cyclical structures. This is particularly interesting for macroeconomic data with a high seasonal component or cyclical movement due to economic activity.

The main problem with the non-linear deterministic trends in the context of fractional integration is that the interaction of the two structures produces a model with a non-linear structure for the coefficients, implying that linear methods are invalid for the estimation of the parameters. Also, a misspecified deterministic component may affect the power of the tests for the order of integration of the variables (see Perron, 1989, amongst many others). Many authors such as Zivot and Andrews (1992), Lumsdaine and Papell (1997), Lee and Strazicich (2003) and Papell and Prodan (2006), *inter alia,* have proposed unit root tests incorporating structural breaks, so as to improve the performance of the tests. However, structural breaks may still not be a proper specification of the deterministic component. Changes can occur smoothly rather than suddenly. In this line, Ouliaris et al. (1989) proposed regular polynomials to approximate deterministic components in the data generation process. However, as later pointed out by Bierens (1997), Chebyshev polynomials might be a better mathematical approximation of the time functions, since Chebyshev polynomials are bounded and orthogonal. Chebyshev polynomials are cosine functions of time, which according to Bierens (1997), can be very flexible to approximate deterministic trends. With respect to the long range dependence we use a very general framework that allows the incorporation of one or more integer or fractional orders of integration of arbitrary order anywhere on the unit circle in the complex plane. This will allow us the analysis of a great variety of model specifications, including for example seasonal and cyclical behaviors of any stationary or nonstationary degree. Also, given that the inference based on t-statistics remains valid under the fractional integration specification used, we propose a very simple way to choose the order of the Chebyshev polynomials based on the significance of the Chebyshev coefficients.

The structure of the paper is as follows: Section 2 describes the statistical model incorporating non-linear (Chebyshev) trends and long range dependence. Section 3 presents a testing procedure for the fractional differencing parameters that includes the estimation of the non-linear trend coefficients. Section 4 contains a simulation study. Section 5 is devoted to the empirical work that includes an application using real effective exchange rates for 40 industrialized countries, and its implications for the purchasing power parity (PPP) theory. Section 6 concludes the paper.

2. The statistical model

We consider the following model,

$$y_t \;=\; f(\theta;\, z_t) \;+\; x_t\,, \qquad t \;=\; 1,\, 2,\, \ldots\,, \tag{1}$$

where y_t is the observed time series, f is a non-linear function that depends on the unknown parameter vector of dimension m, θ, and z_t which is a vector of deterministic terms or weakly exogenous variables; finally, we suppose that the error term x_t can be described in terms of the following model,

$$\rho(L;\, d)\; x_t \;=\; u_t, \qquad t \;=\; 1,\, 2,\, \ldots\,, \tag{2}$$

with

$$\rho(L;\, d) \;=\; (1\;-\;L)^{d_1}\;(1\;+\;L)^{d_2}\prod_{j=3}^{M}(1-2\cos w_r^{(j)}\;L\;+\;L^2)^{d_j} \tag{3}$$

and u_t assumed to be *I(0)*. For the purpose of the present work we define an *I(0)* process as a covariance stationary process with a spectral density function that is positive and bounded at all frequencies in the spectrum. Thus, it includes for u_t in (2) stationary and invertible autoregressive and moving average (*ARMA*) processes. Coming back to (3), L is the backshift operator (i.e., $Lx_t = x_{t-1}$) and d is an ($Mx1$) vector containing the fractional differencing parameters that correspond to different poles or singularities in the spectrum. We observe that this is a very general specification that includes many cases of interest such as the standard *I(d)* models (in case of $d_j = 0$ for all $j \neq 1$, and $d_1 = d$); cyclical fractional models based on Gegenbauer processes (when $d_j = 0$ for all $j \neq 3$); seasonal models ($M = 3$ with $w_r^{(3)} = \pi$), etc. (See Section 3.1 below).

Given the above set-up we focus on the estimation and testing of the unknown parameters corresponding to the vectors d and θ referring respectively to the differencing parameters and the non-linear deterministic trend coefficients.

The main problem we face with this set-up is the interaction between the equations (1) and (2), in particular, between the long memory polynomial ρ and the non-linear function f. Under many circumstances the combination of the two produces a non-linear model in parameters, which hinders the task of estimating the parameter vector θ.

However, one model that accommodates extremely well in the present context is the Chebyshev time polynomial.

The Chebyshev time polynomials $P_{i,T}(t)$ are defined by:

$$P_{0,T}(t) \; = \; 1,$$

$$P_{i,T}(t) \; = \; \sqrt{2}\cos\left(i\,\pi\,(t+0.5)/T\right), \qquad t \; = \; 1, 2, \ldots, T; \quad i \; = \; 1, 2, \ldots \, . \tag{4}$$

See Hamming (1973) for a description of these polynomials. Bierens (1997) uses them in the context of unit root testing. The latter author proposes several unit root tests, which account for a drift and a unit root under the null hypothesis, and stationarity around a linear or non-linear trend under the alternative. Hence, within the analysis of the order of integration of the variables, Bierens (1997) unit root tests, allow us to test whether the process is linear or non-linear trend stationary.

Across the present paper we employ Chebyshev polynomials to describe the deterministic trend. Thus, we can replace (1) by

$$y_t \; = \; \sum_{i=0}^{m} \theta_i \, P_{iT}(t) \; + \; x_t \, , \qquad t \; = \; 1, 2, \ldots , \tag{5}$$

with m indicating the order of the Chebyshev polynomial, and x_t following the model given by (2) and (3). Note that the higher m is the less linear the approximated deterministic component becomes. An issue that immediately arises here is the determination of the optimal choice for m. However, as will be argued below, standard t-statistics will remain valid under the specification given by (5), (2) and (3) noting that the error term is *I(0)* by definition. The choice of m will, then, depend on the significance of the Chebyshev coefficients based on a particular choice of the (possibly *ARMA*) model selected for the *I(0)* disturbances.

3. The procedure

The method proposed in this paper is a slight modification of Robinson (1994). He considers the same set-up as in (1) and (2) with f in (1) of the linear form: $\theta^T z_t$, testing the null hypothesis:

$$H_o : d = d_o, \tag{6}$$

for any real vector value d_o. Under H_o, and using the two equations,

$$y_t^* = \theta^T z_t^* + u_t, \qquad t = 1, 2, \dots, \tag{7}$$

where $y_t^* = \rho(L; d_o) y_t$, and $z_t^* = \rho(L; d_o) z_t$. Then, given the linear nature of the above relationship and the *I(0)* nature of the error term u_t, the coefficients in (7) can be estimated by standard OLS/GLS methods. The same happens in our approach, whereby f contains the Chebyshev polynomials, noting that the relation is linear in parameters. Thus, combining equations (2) and (5) we get

$$y_t^* = \sum_{i=0}^{m} \theta_i P_{iT}^*(t) + u_t, \qquad t = 1, 2, \dots, \tag{8}$$

where

$$P_{iT}^*(t) = \rho(L; d_o) P_{iT}(t),$$

and using OLS/GLS methods, under the null hypothesis (6), the residuals are

$$\hat{u}_t = y_t^* - \sum_{i=0}^{m} \hat{\theta}_i P_{iT}^*(t); \qquad \hat{\theta} = \left(\sum_{t=1}^{T} P_t P_t^T \right)^{-1} \left(\sum_{t=1}^{T} P_t y_t^* \right),$$

and P_t as the (*mx1*) vector of Chebyshev polynomials. Based on the above residuals $\hat{u}_t$, we estimate the variance,

$$\hat{\sigma}^2(\tau) = \frac{2\pi}{T} \sum_{j=1}^{T} g(\lambda_j; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_j); \qquad \lambda_j = 2\pi j / T, \tag{10}$$

where $I_{\hat{u}}(\lambda_j)$ is the periodogram of $\hat{u}_t$; g is a function related with the spectral density of u_t (i.e., s.d.f.(u_t) = $(\sigma^2/2\pi)g(\lambda_j;\tau)$); and the nuisance parameter τ is estimated, for example, by $\hat{\tau} = \arg\min_{\tau \in T^*} \sigma^2(\tau)$, where T^* is a suitable subset of the R^q Euclidean space.

The test statistic, based on Robinson (1994), for testing H_o (6) in (5), (2) and (3) uses the Lagrange Multiplier (LM) principle, and is given by

$$\hat{R} = \frac{T}{\hat{\sigma}^4} \hat{a}^T \hat{A}^{-1} \hat{a}, \tag{11}$$

where T is the sample size, and

$$\hat{a} = \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j)\, g(\lambda_j;\hat{\tau})^{-1} I_{\hat{u}}(\lambda_j),$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^* \psi(\lambda_j)\psi(\lambda_j)^T - \sum_j^* \psi(\lambda_j)\hat{\varepsilon}(\lambda_j)^T \left(\sum_j^* \hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)^T \right) \sum_j^* \hat{\varepsilon}(\lambda_j)\psi(\lambda_j)^T \right)$$

with

$$\psi(\lambda_j) = \mathrm{Re}\left(\frac{\partial}{\partial d} \log \rho(e^{i\lambda_j}; d) \right), \quad \text{and} \quad \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j;\tau)\Big|_{\tau = \hat{\tau}},$$

and the sum over * above refers to all the bounded discrete frequencies in the spectrum. Under very mild regularity conditions[1], Robinson (1994) showed that

$$\hat{R} \;\to_d\; \chi^2_M \qquad as \quad T \to \infty, \tag{12}$$

and, based on Gaussiantiy of u_t, he also showed the Pitman efficiency theory of the test against local departures from the null. That means that if we direct the test against local alternatives of form:

$$H_a: d = d_o + \delta T^{-1/2},$$

[1] These conditions only include moments up to a second order.

where δ is a non-null parameter vector, $\hat{R} \to_d \chi^2_M(\Lambda)$, indicating a non-central chi-squared distribution with non-centrality parameter which is optimal under Gaussiantiy of u_t.

3.1 Simple particular cases

In this section, we simplify the functional form of the above test statistic for some particular cases of interest.

a) White noise u_t

If we suppose that the disturbances are white noise, then, the spectral density function of u_t is simply $\sigma^2/2\pi$, and therefore, $g \equiv 1$. Also, $\hat{\varepsilon}(\lambda_j) = 0$. Then,

$$\hat{a} = \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j) I_{\hat{u}}(\lambda_j), \quad \text{and} \quad \hat{A} = \frac{2}{T}\left(\sum_j^* \psi(\lambda_j)\psi(\lambda_j)^T \right).$$

b) The case of the standard *I(d)* model

A very standard case examined in the literature is the one corresponding to $\rho(L;d) = (1-L)^d$. These processes are called fractionally integrated or I(d); they were introduced by Granger (1980), Granger and Joyeux (1981) and Hosking (1981), and have been widely employed in empirical works in the last twenty years to describe the dynamics of many economic and financial time series (Diebold and Rudebusch, 1989; Sowell, 1992; Gil-Alana and Robinson, 1997; etc.).

In this context, $M = 1$, and $\psi(\lambda_j) = \log \left| 2 \sin \frac{\lambda_j}{2} \right|$, implying that

$$\hat{A} = \frac{2}{T}\sum_{j=1}^{T-1}\left(\log\left|2\sin\frac{\lambda_j}{2}\right|\right)^2,$$

which can be asymptotically approximated by $\pi^2/6$.

c) The case of a cyclical *I(d)* model

In the previous case, the spectral density function is unbounded at the long run or zero frequency. However, the pole or singularity in the spectrum may occur at a non-zero frequency. In such a case we can consider $\rho(L;\, d) = (1 - 2\cos w_r L + L^2)^d$, with $w_r = 2\pi r/T$, $r = T/s$, and thus s will indicate the number of time periods per cycle, while r refers to the frequency that has a pole or singularity in the spectrum of the series. Gray *et al.* (1989, 1994) showed that this polynomial can be expressed in terms of the Gegenbauer polynomial, such that, denoting $\mu = \cos w_r$, for all $d \neq 0$,

$$(1 - 2\mu L + L^2)^{-d} = \sum_{j=0}^{\infty} C_{j,d}(\mu) L^j,$$

where $C_{j,d}(\mu)$ are orthogonal Gegenbauer polynomial coefficients defined recursively as:

$$C_{0,d}(\mu) = 1, \qquad C_{1,d}(\mu) = 2\mu d,$$

$$C_{j,d}(\mu) = 2\mu\left(\frac{d-1}{j}+1\right)C_{j-1,d}(\mu) - \left(2\frac{d-1}{j}+1\right)C_{j-2,d}(\mu), \quad j = 2, 3, \ldots,$$

(see Magnus et al., 1966, Rainville, 1960, etc. for further details on Gegenbauer polynomials). This type of process was introduced by Andel (1986) and subsequently analysed by Gray, Zhang and Woodward (1989, 1994), Chung (1996a,b), Gil-Alana (2001) and Dalla and Hidalgo (2005) among many others.

In this case, M is also equal to 1, and

$$\psi(\lambda_j) \;=\; \log\left(\cos\lambda_j \;-\; \cos w_r\right).$$

d) The case of multiple cycles

We can also study the case of processes that contain multiple poles or singularities in the spectrum. In these cases, $\rho(L;d) = \prod_{u=1}^{M}(1-2\cos w_r^{(u)} L + L^2)^{d_u}$. These processes were introduced by Giraitis and Leipus (1995), Woodward et al. (1998), Ferrara and Guegan (2001), and Sadek and Khotanzad (2004) among others. One special case here is the seasonal *I(d)* model that, using a very simple specification may be expressed as

$$(1 \;-\; L^s)^d \;x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots,$$

s indicating the number of time periods per year. Thus, for example, for quarterly data, *s* = 4, and it is a particular case of *d*) with M = 3, and $w_r^{(u)} =$ *0, π/2* and *π* respectively for (*u)* = 1, 2 and 3. These processes were introduced by Porter-Hudak (1990) and have been subsequently examined by Ray (1993), Sutcliffe (1994) and Gil-Alana and Robinson (2001) and others.

If $s = 4$ and $\rho(L;\, d) = (1 - L^4)^d$, then $M = 1$,[2] and $\psi(\lambda_j)$ becomes:

$$\psi(\lambda_j) \;=\; \log\left|2\sin\frac{\lambda_j}{2}\right| \;+\; \log\left(2\cos\frac{\lambda_j}{2}\right) +\; \log\left|2\,\cos\lambda_j\right|,$$

and allowing for a greater degree of generality, we can consider the case of different orders of integration at each frequency, so that $\rho(L;d) \;=\; (1-L)^{d_1}(1+L)^{d_2}(1-L^2)^{d_3}$. In this case, $M = 3$ and $\psi(\lambda_j)$ becomes a (*3x1*) vector of form:

[2] Note that M refers to the dimension of *d*, while m indicates the order of the Chebyshev polynomials.

$$\left(\log \left| 2\sin\frac{\lambda_j}{2} \right|; \log\left(2\cos\frac{\lambda_j}{2} \right); \log\left| 2\cos\lambda_j \right| \right)^T.$$

e) **The case of Bloomfield (1973) disturbances**

Finally, we can suppose that the disturbances u_t follow a non-parametric approach due to Bloomfield (1973). This model does not provide an explicit formula for the error term, but it is implicitly determined by its spectral density function, which is given by

$$f(\lambda_j;\tau) = \frac{\sigma^2}{2\pi} \exp\left(2\sum_{r=1}^{X} \tau_r \cos(\lambda_j\, r) \right), \tag{13}$$

where X indicates the number of parameters required to describe the short run dynamics. Bloomfield (1973) showed that the logarithm of an estimated spectral density function is often found to be a fairly well behaved function and thus can be approximated by a truncated Fourier series. He showed that (13) approximates the spectral density of an *ARMA(p, q)* process well when p and q are small values, which is usually the case for most economic time series. Like the stationary *AR* model, this has exponentially decaying autocorrelations and thus, using this specification, one does not need to rely on as many parameters as in the case of *ARMA* processes. Moreover, it accommodates extremely well in the context of the testing procedure presented above. Thus, formulae for Newton-type iterations for estimating the τ_j are very simple (involving no matrix inversion), updating formulae when X is increased is also simple, and we can replace $\hat{A}$ in the functional form of the test statistic in (11) by the population quantity:

$$\sum_{l=X-1}^{\infty} l^{-2} = \frac{\pi^2}{6} - \sum_{l=1}^{X} l^{-2},$$

which indeed is constant with respect to the τ_j.[3]

4. A simulation experiment

In this section we briefly examine the finite sample behavior of some simple versions of the tests by means of Monte Carlo simulations. All calculations were carried out using Fortran and the programs are available from the authors upon request. Given the variety of cases and the number of possibilities covered by the tests, we concentrate on some simple cases, widely employed in the literature such as the case of standard *I(d)* processes with the singularity or pole in the spectrum occurring at the long run or zero frequency. In particular, we consider the following data generation process (DGP):

$$y_t = \sum_{i=0}^{m} \theta_i P_{iT}(t) + x_t, \qquad (1 - L)^d x_t = u_t, \tag{14}$$

with m = 3 to justify some degree of non-linear behavior, and u_t as a white noise process with mean zero and variance 1. Also, for simplicity, we suppose that $\theta_i = 1$ for all i, and take d in (14) equal to 0, 0.25, 0.50, 0.75 and 1, thus, including stationary and nonstationary hypotheses. We generate Gaussian series using the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986), for different sample sizes T = 50, 100, 300 and 500, taking 10,000 replications for each case, and present the results for a nominal size of 5%.

[3] See Gil-Alana (2004) for an explanation of the accommodation of the model of Bloomfield (1973) in the context of fractional integration, and more in particular, in the context of the tests of Robinson (1994).

Based on the model given by (14) we test the null hypothesis (6) for different d_o-values. However, noting that in this context *M = 1*, we can consider one–sided alternatives such as H_a*:* $d > d_o$ or $d > d_o$, and then, consider the test statistic:

$$\hat{r} \;=\; \sqrt{\hat{R}} \;=\; \frac{\sqrt{T}}{\sqrt{\hat{A}}}\frac{\hat{a}}{\hat{\sigma}^2}, \tag{15}$$

which is asymptotically distributed as

$$\hat{r} \;\rightarrow_d\; N(0,1) \qquad as \quad T \rightarrow \infty, \tag{16}$$

See Robinson (1994). Thus, an approximate one-sided 100α%-level of (6) against the alternative $d > d_o$ is given by the rule:

"Reject H_o *if* $\hat{r} > z_\alpha$*",*

where the probability that a standard normal variate exceeds z_α is α. In the same way, an approximate one-sided 100α%-level of (6) against the alternative $d < d_o$ is given by the rule:

"Reject H_o *if* $\hat{r} < -z_\alpha$*".*

We examine the size and the power properties of the test in the case of the model given by (14) with $d = 1$ and look in Table 1 at the rejection frequencies of $\hat{r}$ in (15) with d_o = 0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75 and 2. Thus, the values corresponding to d_o =1 will indicate the size of the test. We see in this table that the sizes of the tests are clearly biased if the sample size is small. Thus, for example, if T = 50 and the tests are directed against $d > d_o$, the size is 0.018; however, when directed against $d < d_o$, it becomes much higher than the nominal size of 0.050 (0.109); however, as the sample size increases the values tend to approximate to the 5% level, which is consistent with the asymptotic nature of the tests. If we focus now on the rejection frequencies, we observe that the higher sizes observed in the case of $d < d_o$ also produce higher rejection probabilities in all cases compared with the case of alternatives with $d < 1$. Nevertheless, for departures higher than 0.5 even with small sample sizes, the tests behave fairly well, and if $T \geq 300$ the probabilities are very close to 1 in all cases. Remember here that the null consists of a unit root with Chebyshev polynomials, so the test performs well even in strong nonstationary contexts. Performing the experiment with θ-coefficients different from 1, and also with other values of d lead to essentially the same conclusions implying that the test performs relatively well if the sample size is large enough.

[Insert Tables 1 and 2 about here]

Next we perform a similar experiment in non-Gaussian contexts. For this purpose, we examine the same null model as in Table 1 but assuming now that the disturbances are *t*-Student distributed with 3 degrees of freedom. This distribution is interesting because it just satisfies the second moment condition required in the test, its third moments not existing. The results, displayed in Table 2, are competitive with the Gaussian ones, with the sizes being closer to the nominal one of 5% in practically all cases. If we focus on the rejection frequencies, they tend to be slightly larger for values of $d_o < 1$, and lower when $d_o > 1$ compared with Table 1. Very similar results were obtained if weak autocorrelation is permitted for the *I(0)* disturbances term, and the same applies for other values of d in (14).

5. An empirical application

In this section we apply the fractional integration tests developed in this paper to examine the mean reversion of real exchange rates and purchasing power parity (PPP). The absolute version of the PPP theory postulates that the price levels in two different countries should converge when measured in the same currency, so as to equalize the purchasing power of the currencies. This, therefore, implies that the real exchange rate, defined as the ratio of prices in both places, translated to a common currency using the nominal exchange rate, should converge to 1. However, it is well known within the literature that the absolute version of the PPP hypothesis may be too restrictive. Hence, a less restrictive version of PPP is the relative PPP hypothesis, which implies that prices in common currency may converge to a constant different from 1. This relative version of the PPP implies then that what is actually expected in the long run is that the real exchange rate should be reverting to a constant, which may be different from 1. The intuition behind this is related to the fact that because of the existence of trade barriers, transport costs, and different measures of price indices, there may be a gap between price levels in different countries. Hence, on average, changes in real exchange rates should be zero, according to the relative version of the PPP theory.

In view of the above comments, testing for mean reversion becomes of paramount importance when testing for the empirical validity of the PPP theory, which at the same time, can be seen as a measure of the degree of over/under-valuation of the currencies, and it is used as a base for a number of macroeconomic models, i.e. the Dornbusch model. However, real exchange rate convergence, on average, to a constant along time may not be very realistic, in particular when countries experience different levels of economic growth and productivity gains, as well as, when countries suffer from changes in economic fundamentals, which may indeed change the equilibrium value of real exchange rates. For instance, the well known dynamic Penn effect and the Balassa-Samuelson effect, may induce deterministic trends in the data (see Lothian and Taylor, 2000, among others), and the existence of structural changes, may, in addition, induce changes in those trends. Hence, the importance of controlling for non-linear deterministic trends when testing for real exchange rate mean reversion. In a recent contribution, Cushman (2008) tests for the PPP hypothesis using the Bierens (1997) unit root tests for bilateral exchange rates. He finds evidence to support that real exchange rates may in fact contain non-linear trends. However, it is not possible to test for the significance of these trends, unless the null is rejected.

Our newly developed fractional integration testing procedure, taking into account Chebyshev polynomials to approximate non-linear deterministic trends, solves these problems with the flexibility of having non-integer orders of integration. Given that the residuals of the auxiliary regression are *I(0)* stationary by assumption, *t*-statistics are valid to test for the significance of the non-linear trends. This novelty solves the problem of choosing the order of the Chebyshev polynomials, which was not clearly defined by Bierens (1997).

The data used in the empirical application are real effective exchange rates against each country's 27 main trade partners, downloaded from *Eurostat* (code *ert_eff_ic_q*) for 40 countries, with different degrees of economic integration and development. We have used quarterly data from 1994:Q1 until 2011:Q3.

Across this section we consider the following model,

$$y_t \;=\; \sum_{i=0}^{m} \theta_i \, P_{iT}(t) \;+\; x_t\,, \qquad (1 \;-\; L)^d \, x_t \;=\; u_t\,, \tag{17}$$

assuming that u_t is a white noise process.

Table 3 displays the estimates of d and the 95% confidence bands of the non-rejection values of d for the cases of m = 0, 1, 2 and 3. Higher values of m lead to non-significant coefficients for θ_i in all cases. These estimates were obtained using the Whittle function in the frequency domain and they coincide with the values of d_o that produce the lowest statistics in absolute value when using our testing approach with a fine grid of d_o-values (with 0.001 increments). We observe in this table that the values of *d* are very similar across the different values for *m*, in general, observing a slight reduction in the degree of integration as we increase *m.*[4] We also notice that most of the estimates of *d* are within the unit root interval and some of them are even significantly above 1. The only evidence of mean reversion (i.e. d significantly below 1) is obtained for the cases of Cyprus, Greece and Malta (for all values of *m*) and for France and Spain if *m* = 2 or 3, i.e. assuming the existence of non-linearities. The results from Table 3 also point out that it is possible to reduce the order of integration of the variable by increasing artificially the order of the Chebyshev polynomials, *m*. This is consistent with other works that show that fractional integration and nonlinearities are issues which are intimately related (Diebold and Inoue, 2001; Granger and Hyung, 2004; etc.).

[Insert Tables 3 – 5 about here]

Next we examine the deterministic terms in more detail, checking if the Chebyshev coefficients are statistically significant for the selected estimates of d. The results are presented in Table 4. We notice several cases where non-linearities are present. Based on these significant terms, we selected the appropriate model for each series, and the summary of the results (based only on the significant Chebyshev coefficients) are reported in Table 5. We see that strong evidence of non-linearities (with the two non-linear coefficients statistically significantly different from zero) is obtained for the cases of Cyprus, France, Malta, Spain, Germany, Hong-Kong and Lithuania. In the first four cases, the unit root hypothesis is rejected in favour of mean reversion, while in the

[4] This might indicate a degree of competition between the non-linear structure due to the Chebyshev polynomials and the *I(d)* framework in describing the structure of the series.

remaining three cases, though the estimated values of *d* are smaller than 1, the unit root cannot be rejected. Evidence of non-linearity with significant θ_2-coefficient is observed for Austria, Greece and Slovakia, the unit root being rejected in favor of mean reversion in the case of Greece. Also, for some countries only one of the two non-linear coefficients is significant, such as China (with only θ_3 being statistically significant, and an estimate of d of 0.979) as well as Bulgaria and Latvia (with d equal to 0.827 and 1.197 respectively), and also, Belgium, Brazil and the UK (with θ_2 significant but not θ_3) and the unit root being not rejected. For the remaining cases, only an intercept or a linear trend is required.

We also conducted the analysis based on weakly autocorrelated errors. We tried both seasonal and non-seasonal autoregressions and the results, not displayed, indicate that though quantitatively there are some differences when computing the results based on autocorrelated errors qualitatively the same conclusions hold, since the number of cases corresponding to "mean reversion", "unit roots" or "explosive roots" affect exactly to the same series as in the case of white noise errors.

Our results pinpoint a few economic insights. We first observe that in most cases structural breaks in the form of non-linear trends are present in the data. Second, for a number of countries, for instance the Czech Republic and Hungary, a linear trend is enough to approximate the data. This implies that the Balassa-Samuelson effect might be present, which makes economic sense given the process of catching-up with Western Europe during the transition period from communism to market economies. Finally, that in all cases of mean reversion, it occurs along with structural breaks. Comparing our results to those by Cushman (2008), although the results are not directly comparable, we can say that we find evidence of mean reversion using a lower order for the Chebyshev polynomials.

5. Concluding comments

In this paper we have examined a model that incorporates Chebyshev polynomials in time in the context of long range dependence. For the latter we use a very general expression that permits us to examine stationary and nonstationary hypotheses with one or more unit or fractional degrees of integration with the singularities in the spectrum occurring at zero and non-zero frequencies. The main advantage of this model is that combining the two structures (non-linear Chebyshev polynomials and fractional integration) leads to a new model that is linear in parameters, permitting the estimation of the Chebyshev polynomials in a very simple way. Moreover, we describe a testing procedure, originally proposed by Robinson (1994) that displays several advantages in the present context. Thus, it allows us to test any real vector value for the differencing parameters, including stationary and nonstationary hypotheses; the incorporation of the Chebyshev polynomials allows its estimation with a straightforward method, including the use of the significancy of the coefficients throughout standard *t*-values. The limit distribution of the procedure is standard chi-squared distributed, and several Monte Carlo experiments conducted in the paper show it performs well even with small samples. A small empirical application based on this approach and using real effective exchange rates is also conducted in the paper.

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Table 1: Rejection frequencies against one-sided alternatives with Gaussian u_t

	d_o	T = 50	T = 100	T = 300	T = 500
H_a: d > d_o	0.00	0.788	0.907	1.000	1.000
	0.25	0.519	0.788	0.903	0.999
	0.50	0.308	0.554	0.702	0.945
	0.75	0.103	0.341	0.671	0.893
	1.00	**0.018**	**0.027**	**0.039**	**0.047**
H_a: d < d_o	**1.00**	**0.109**	**0.088**	**0.075**	**0.056**
	1.25	0.608	0.701	0.855	0.939
	1.50	0.771	0.886	0.996	0.998
	1.75	0.983	1.000	1.000	1.000
	2.00	1.000	1.000	1.000	1.000

The nominal size is 5%. In bold the size of tests.

Table 2: Rejection frequencies against one-sided alternatives with t_3-distributed u_t

	d_o	T = 50	T = 100	T = 300	T = 500
H_a: d > d_o	0.00	0.793	0.914	1.000	1.000
	0.25	0.520	0.793	0.955	1.000
	0.50	0.311	0.570	0.724	0.946
	0.75	0.107	0.344	0.683	0.894
	1.00	**0.022**	**0.034**	**0.040**	**0.047**
H_a: d < d_o	**1.00**	**0.101**	**0.088**	**0.069**	**0.055**
	1.25	0.603	0.693	0.831	0.917
	1.50	0.747	0.877	0.974	0.981
	1.75	0.979	0.992	1.000	1.000
	2.00	1.000	1.000	1.000	1.000

The nominal size is 5%. In bold the size of tests.

Table 3: Estimates of d based on white noise disturbances

Series	$m = 0$	$m = 1$	$m = 2$	$m = 3$
AUSTRIA	1.055 (0.907, 1.272)	1.048 (0.907, 1.260)	0.971 (0.786, 1.224)	0.929 (0.711, 1.197)
AUSTRALIA	1.205 (1.007, 1.501)	1.209 (1.021, 1.493)	1.202 (0.997, 1.497)	1.199 (0.984, 1.493)
BELGIUM	1.206 (1.069, 1.395)	1.203 (1.068, 1.391)	1.137 (0.955, 1.355)	1.123 (0.939, 1.337)
BRAZIL	1.114 (0.961, 1.356)	1.103 (0.952, 1.341)	1.029 (0.832, 1.317)	0.986 (0.746, 1.303)
BULGARY	0.914 (0.743, 1.261)	0.948 (0.737, 1.263)	0.947 (0.734, 1.266)	0.821 (0.451, 1.222)
CANADA	1.136 (0.895, 1.465)	1.133 (0.884, 1.462)	1.133 (0.882, 1.461)	1.121 (0.863, 1.461)
CHINA	1.179 (1.025, 1.426)	1.168 (1.022, 1.409)	1.160 (1.016, 1.407)	0.953 (0.694, 1.317)
CYPRUS	**0.658** **(0.568, 0.806)**	**0.602** **(0.478, 0.784)**	**0.503** **(0.347, 0.725)**	**0.429** **(0.242, 0.688)**
CZECK REP.	1.003 (0.806., 1.389)	1.049 (0.822, 1.392)	1.041 (0.802, 1.394)	0.972 (0.633, 1.384)
DENMARK	1.058 (0.861, 1.323)	1.062 (0.860, 1.327)	1.048 (0.847, 1.316)	1.048 (0.830, 1.322)
ESTONIA	1.439 (1.274, 1.681)	1.443 (1.293, 1.667)	1.443 (1.295, 1.673)	1.399 (1.252, 1.617)
FINLAND	1.202 (0.993, 1.495)	1.190 (0.974, 1.486)	1.179 (0.953, 1.486)	1.176 (0.951, 1.473)
FRANCE	0.907 (0.821, 1.043)	0.859 (0.748, 1.016)	**0.721** **(0.556, 0.933)**	**0.664** **(0.453, 0.897)**
GERMANY	1.072 (0.941, 1.255)	1.072 (0.940, 1.255)	0.985 (0.825, 1.200)	0.935 (0.747, 1.164)
GREECE	**0.774** **(0.661, 0.933)**	**0.800** **(0.685, 0.954)**	**0.722** **(0.569, 0.904)**	**0.701** **(0.543, 0.897)**
HONG-KONG	1.206 (1.067, 1.425)	1.187 (1.032, 1.414)	1.158 (1.002, 1.396)	0.987 (0.741, 1.293)
HUNGARY	0.909 (0.753, 1.344)	0.759 (0.427, 1.307)	0.755 (0.405, 1.307)	0.738 (0.384, 1.307)
IRELAND	1.195 (1.035, 1.437)	1.149 (0.972, 1.403)	1.148 (0.973, 1.404)	1.114 (0.907, 1.396)
ITALY	1.062 (0.833, 1.352)	1.062 (0.825, 1.349)	1.059 (0.817, 1.343)	1.050 (0.807, 1.331)
JAPAN	1.067 (0.904, 1.311)	1.056 (0.884, 1.305)	1.056 (0.881, 1.306)	1.027 (0.851, 1.283)
LATVIA	1.326 (1.167, 1.553)	1.336 (1.193, 1.554)	1.333 (1.192, 1.554)	1.193 (0.992, 1.462)
LITUANIA	1.146 (1.013, 1.322)	1.184 (1.081, 1.333)	1.146 (1.037, 1.306)	0.941 (0.766, 1.177)
MALTA	**0.694** **(0.584, 0.938)**	**0.738** **(0.606, 0.975)**	**0.676** **(0.523, 0.920)**	**0.309** **(0.071, 0.693)**
MEXICO	1.151 (1.003, 1.352)	1.150 (1.003, 1.351)	1.116 (0.952, 1.334)	1.097 (0.932, 1.308)

(continued)

NETHERLANDS	1.088 (0.932, 1.304)	1.082 (0.924, 1.301)	1.081 (0.922, 1.303)	1.034 (0.861, 1.262)
NORWAY	0.944 (0.729, 1.222)	0.952 (0.741, 1.222)	0.949 (0.731, 1.229)	0.943 (0.722, 1.221)
NEW ZEELAND	1.274 (1.039, 1.624)	1.265 (1.044, 1.603)	1.264 (1.034, 1.603)	1.255 (1.033, 1.566)
POLAND	1.029 (0.804, 1.354)	1.034 (0.823, 1.346)	1.023 (0.796, 1.346)	0.997 (0.741, 1.331)
PORTUGAL	1.051 (0.872, 1.292)	1.039 (0.855, 1.288)	1.039 (0.853, 1.288)	0.984 (0.762, 1.244)
ROMANIA	1.145 (0.931, 1.433)	1.134 (0.917, 1.433)	1.133 (0.917, 1.435)	1.129 (0.906, 1.422)
RUSSIAN FED.	1.245 (1.022, 1.533)	1.249 (1.042, 1.553)	1.242 (1.027, 1.556)	1.242 (1.027, 1.554)
SOUTH KOREA	1.094 (0.861, 1.411)	1.096 (0.873, 1.417)	1.096 (0.876, 1.417)	1.083 (0.846, 1.407)
SLOVAKIA	1.137 (0.984, 1.417)	1.107 (0.944, 1.395)	0.992 (0.722, 1.366)	0.980 (0.692, 1.354)
SLOVENIA	1.342 (1.037, 1.755)	1.337 (1.072, 1.744)	1.342 (1.077, 1.711)	1.332 (1.054, 1.591)
SPAIN	0.907 (0.813, 1.047)	0.859 (0.744, 1.016)	**0.721** **(0.554, 0.933)**	**0.664** **(0.459, 0.899)**
SWEDEN	1.063 (0.854, 1.376)	1.044 (0.807, 1.377)	1.044 (0.803, 1.364)	1.042 (0.797, 1.382)
SWITZERLAND	1.252 (1.096, 1.463)	1.207 (1.076, 1.398)	1.208 (1.073, 1.384)	1.207 (1.066, 1.373)
TURKEY	0.824 (0.643, 1.308)	0.677 (0.318, 1.269)	0.678 (0.317, 1.263)	0.648 (0.207, 1.255)
U.K.	1.227 (1.082, 1.433)	1.228 (1.087, 1.444)	1.151 (0.956, 1.405)	1.132 (0.933, 1.388)
U.S.A	1.212 (1.047, 1.346)	1.213 (1.045, 1.467)	1.172 (0.983, 1.444)	1.119 (0.911, 1.393)

Note: In bold, evidence of mean reversion ($d < 1$). In brackets we display the confidence intervals at the 95%.

Table 4: Estimates of the Chebyshev polynomials in the case of *m* = 3

Series	θ_0	θ_1	θ_2	θ_3
AUSTRIA	**99.016 (39.39)**	**2.373 (1.68)**	**1.575 (2.01)**	0.707 (1.31)
AUSTRALIA	**107.251 (3.10)**	-3.701 (-0.17)	4.266 (0.49)	-1.872 (-0.35)
BELGIUM	**99.301 (20.94)**	0.610 (0.21)	**2.286 (1.81)**	0.665 (0.83)
BRAZIL	**102.362 (3.09)**	6.446 (0.33)	**18.811 (1.90)**	-8.191 (-1.23)
BULGARY	**114.558 (11.12)**	**-27.761 (-4.75)**	-1.164 (-0.32)	**-5.309 (-2.06)**
CANADA	**126.198 (4.82)**	-7.224 (-0.45)	-1.048 (-0.15)	-2.934 (-0.66)
CHINA	**90.971 (6.95)**	0.391 (0.05)	-3.134 (-0.77)	**-9.764 (-3.56)**
CYPRUS	**104.742 (130.10)**	**-2.909 (-5.77)**	**1.089 (2.60)**	**0.613 (1.70)**
CZECK REP.	**115.654 (9.51)**	**-21.675 (-3.03)**	1.994 (0.54)	-3.420 (-1.37)
DENMARK	**99.238 (21.86)**	-0.806 (-0.29)	0.802 (0.62)	0.020 (0.02)
ESTONIA	**67.849 (2.65)**	-0.254 (-0.38)	0.651 (0.12)	-2.962 (-0.96)
FINLAND	**93.879 (9.22)**	2.774 (0.43)	1.587 (0.61)	0.490 (0.30)
FRANCE	**104.771 (91.91)**	**-4.643 (-7.21)**	**1.555 (3.40)**	**0.545 (1.67)**
GERMANY	**97.290 (26.80)**	**4.052 (1.91)**	**2.630 (2.31)**	**1.188 (1.72)**
GREECE	**99.848 (43.86)**	**-3.694 (-2.88)**	**1.791 (2.03)**	-0.690 (-1.02)
HONG-KONG	**87.038 (6.70)**	**12.242 (1.90)**	**-6.114 (-1.87)**	**-8.055 (-3.09)**
HUNGARY	**120.065 (18.02)**	**-17.302 (-4.61)**	-0.560 (-0.22)	1.031 (0.55)
IRELAND	**107.079 (13.75)**	-5.871 (-1.23)	0.080 (0.03)	1.548 (1.16)
ITALY	**103.153 (13.02)**	-2.892 (-0.60)	-0.518 (-0.23)	-0.810 (-0.55)
JAPAN	**102.040 (4.28)**	10.698 (0.75)	-0.514 (-0.07)	-4.812 (-1.06)
LATVIA	**92.662 (5.18)**	**-13.363 (-1.20)**	-1.255 (-0.28)	**-7.690 (-2.78)**
LITUANIA	**99.697 (11.62)**	**-21.677 (-4.33)**	**-6.914 (-2.58)**	**-7.910 (-4.32)**
MALTA	**101.467 (136.29)**	**-5.077 (-9.78)**	**-2.017 (-4.42)**	**-2.416 (-5.88)**
MEXICO	**141.923 (3.47)**	-2.875 (-0.11)	-13.782 (-1.24)	-7.252 (-1.01)

(continued)

NETHERLANDS	**102.410** **(21.65)**	-1.291 (-0.45)	0.254 (0.18)	1.339 (1.50)
NORWAY	**104.085** **(11.81)**	-1.934 (-0.37)	0.741 (0.27)	-0.730 (-0.39)
NEW ZEELAND	**87.099** **(2.01)**	7.085 (0.26)	2.264 (0.22)	3.277 (0.52)
POLAND	**108.376** **(5.03)**	-11.451 (-0.90)	-3.565 (-0.55)	-3.853 (-0.90)
PORTUGAL	**101.431** **(34.79)**	**-3.058** **(-1.78)**	-0.064 (-0.07)	0.906 (1.54)
ROMANIA	**119.977** **(3.29)**	-22.595 (-1.01)	-1.933 (-0.20)	-1.976 (-0.32)
RUSSIAN FED.	99.158 (0.93)	-13.008 (-0.19)	12.455 (0.49)	-1.812 (-0.11)
SOUTH KOREA	**110.881** **(3.08)**	6.752 (0.31)	-1.080 (-0.10)	3.805 (0.57)
SLOVAKIA	**131.870** **(10.30)**	**-34.771** **(-4.61)**	**7.797** **(2.02)**	-1.482 (-0.57)
SLOVENIA	**86.649** **(5.25)**	-0.551 (-0.05)	0.370 (0.09)	-0.693 (-0.32)
SPAIN	**104.771** **(91.91)**	**-4.643** **(-7.21)**	**1.554** **(3.40)**	**0.545** **(1.73)**
SWEDEN	**95.991** **(8.10)**	4.633 (0.65)	0.266 (0.07)	-0.599 (-0.27)
SWITZERLAND	**93.915** **(4.87)**	8.133 (0.67)	0.222 (0.04)	-0.632 (-0.21)
TURKEY	**111.308** **(11.74)**	**-16.485** **(-3.07)**	0.170 (0.04)	-1.649 (-0.54)
U.K.	**97.291** **(5.72)**	2.831 (0.27)	**-8.440** **(-1.88)**	-2.380 (-0.84)
U.S.A	**112.153** **(4.76)**	1.777 (0.12)	-8.525 (-1.35)	-5.725 (-1.43)

In bold, significant coefficients at the 5% level. T-statistics are given in brackets.

Table 5: Summary results based on the selected model for each series

Series	d	θ_0	θ_1	θ_2	θ_3
AUSTRIA	**0.971 (UR)**	**99.839**	**2.464**	**1.578**	**---**
AUSTRALIA	**1.205 (AB)**	**105.363**	**---**	**---**	**---**
BELGIUM	**1.138 (UR)**	**101.068**	**---**	**2.307**	**---**
BRAZIL	**1.043 (UR)**	**99.724**	**---**	**18.879**	**---**
BULGARIA	**0.827 (UR)**	**113.038**	**-27.794**	**---**	**-5.316**
CANADA	**1.136 (UR)**	**110.355**	**---**	**---**	**---**
CHINA	**0.979 (UR)**	**87.019**	**---**	**---**	**-9.745**
CYPRUS	**0.429 (MR)**	**104.742**	**-2.909**	**1.089**	**0.613**
CZECK REP.	**1.049 (UR)**	**112.905**	**-21.855**	**---**	**---**
DENMARK	**1.058 (UR)**	**99.257**	**---**	**---**	**---**
ESTONIA	**1.439 (AB)**	**54.584**	**---**	**---**	**---**
FINLAND	**1.202 (UR)**	**100.749**	**---**	**---**	**---**
FRANCE	**0.664 (MR)**	**104.771**	**-4.643**	**1.555**	**0.545**
GERMANY	**0.935 (UR)**	**97.290**	**4.052**	**2.630**	**1.188**
GREECE	**0.722 (MR)**	**99.245**	**-3.901**	**1.781**	**---**
HONG KONG	**0.987 (UR)**	**87.038**	**12.242**	**-6.114**	**-8.055**
HUNGARY	**0.759 (UR)**	**120.406**	**-17.042**	**---**	**---**
IRELAND	**1.195 (AB)**	**101.113**	**---**	**---**	**---**
ITALY	**1.062 (UR)**	**97.160**	**---**	**---**	**---**
JAPAN	**1.067 (UR)**	**109.704**	**---**	**---**	**---**
LATVIA	**1.197 (AB)**	**90.952**	**-13.409**	**---**	**-7.681**
LITUANIA	**0.941 (UR)**	**99.697**	**-21.677**	**-6.914**	**-7.910**
MALTA	**0.309 (MR)**	**101.46**	**-5.077**	**-2.017**	**-2.416**
MEXICO	**1.151 (AB)**	**108.404**	**---**	**---**	**---**
NETHERLANDS	**1.088 (UR)**	**102.809**	**---**	**---**	**---**
NORWAY	**0.944 (UR)**	**101.390**	**---**	**---**	**---**
NEW ZEELAND	**1.274 (AB)**	**105.200**	**---**	**---**	**---**
POLAND	**1.029 (UR)**	**81.641**	**---**	**---**	**---**
PORTUGAL	**1.039 (UR)**	**102.489**	**-2.993**	**---**	**---**
ROMANIA	**1.145 (UR)**	**82.272**	**---**	**---**	**---**
RUSSIAN FED.	**1.245 (AB)**	**95.927**	**---**	**---**	**---**
SOUTH KOREA	**1.094 (UR)**	**124.409**	**---**	**---**	**---**
SLOVAKIA	**0.992 (UR)**	**129.793**	**-34.799**	**7.812**	**---**
SLOVENIA	**1.342 (AB)**	**85.759**	**---**	**---**	**---**
SPAIN	**0.664 (MR)**	**104.771**	**-4.643**	**1.554**	**0.545**
SWEDEN	**1.063 (UR)**	**102.116**	**---**	**---**	**---**
SWITZERLAND	**1.252 (AB)**	**104.935**	**---**	**---**	**---**
TURKEY	**0.677 (UR)**	**110.839**	**-16.902**	**---**	**---**
U.K.	**1.152 (UR)**	**98.004**	**---**	**-8.481**	**---**
U.S.A.	**1.212 (AB)**	**94.615**	**---**	**---**	**---**

MR means Mean Reversion ($d < 1$) and AB refers to the cases where d is significantly greater than 1. UR means that it contains the unit root case (i.e. $d = 1$).