On a relationship between distorted and spectral risk measures

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Abstract

We study the relationship between two widely used risk measures, spectral measures and distortion risk measures. In both cases, the risk measure can be thought of as a re-weighting of some initial distribution. We prove that spectral risk measures are equivalent to distorted risk pricing measures, or equivalently, spectral risk functions are equivalent to distortion functions. Besides, we prove that distorted measures are absolutely continuous with respect to the original measure. This allows us to find a link between the risk measures based on relative entropy and spectral risk measures or measures based on distortion risk functions.

Key words. Coherent risk measure, Distortion function, Spectral measures, risk aversion function.

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1 Introduction

The quantification of market risk for derivative pricing, for portfolio optimization and pricing risk for insurance purposes has generated a large amount of theoretical and practical work, with a variety of interconnections.

Two lines of research in these areas are based upon, and use as point of departure axioms that both, the market risk measure and the risk pricing measure, have to satisfy.

Value at Risk (VaR) is one of the most popular risk measures, due to its simplicity. VaR indicates the minimal loss incurred in the worse outcomes a portfolio. But this risk measure is not always sub-additive, nor convex. To overcome this, Artzner, Delbaen, Ebner and Heath (1999) proposed the main properties that a risk measures must satisfy, thus establishing the notion of coherent risk measure.

After coherent risk measures and their properties were established, other classes of measures have been proposed, each with distinctive properties: convex (Föllmer and Shied, 2004), spectral (Acerbi, 2002) or deviation measures (Rockafellar et al. 2006).

The coherent risk measures have been used for capital allocation and portfolio optimization as in Rockafellar, Uryasev and Zabarankin (2002), as well as to price options in incomplete markets, as in Cherny (2006).

Spectral risk measures are coherent risk measures that satisfy two additional conditions. These measures have been applied to futures clearinghouse margin requirements in Cotter and Dowd (2006). Acerbi and Simonetti (2002) extend the results of Pflug and Rockafellar-Uryasev methodology to spectral risk measures.

A description of the axioms of risk pricing measures with many applications to insurance can be found in Wang, Young and Panjer (1997), and in the monograph by Kass, Goovaerts, Dhaene and Denuit (2001). The concept of distorted risk measures evolved from this line of work and ties in with the notion of capacity. Capacities are non-additive, monotone set functions which extend the notion of integral in a peculiar way. The evolution of this concept, from Choquet's work in the 1950's until the 1990's can be traced back from the review by Denneberg (1997).

Interestingly enough, there have been some natural points of contact between actuarial and financial risk theory. On the one hand, concepts in actuarial risk

theory can be used to solve problems in derivative pricing, and vice versa. A few papers along these lines are those by Embrechts (1996), Gerber and Shiu (2001), Schweitzer (2001), Goovaerts and Laeven (2006) and Madan and Unal (2004).

Very many risk measures are proposed in the literature, their differences lie in the properties they satisfy. It is very interesting to study the equivalence between these risk measures. It is the purpose of this note to establish an equivalence between spectral risk measures and distorted risk pricing measures. Then we shall examine some other way of computing distorted measures.

This paper is organized as follows: in Section 2 we introduce the concept of coherent and spectral risk measures as well as that of a distortion measure. We present different examples of these measures. In Section 3 we study the equivalence between spectral risk measures and coherent distorted risk measures. For that we present a simpler mathematical proof of the established relationship between spectral risk measures and distortion risk measures. Our proof extends some what the results obtained by Pflug (2004) and Föllmer and Schied (2004), who consider bounded risks. In the process we relate the risk spectrum to the distortion function. In Section 4 we further examine the nature of the distorted distribution function and the relationship between the distorted distributions of different investors, finding a relationship between the measures studied in the Section 3 and the measures based on relative entropy. Finally, Section 5 concludes the paper.

2 Preliminaries

We shall consider a one period market model $(\Omega, \mathcal{F}, P)$. The information about the market, that is the σ-algebra $\mathcal{F}$, can be assumed to be generated by a finite collection of random variables, i.e., $\mathcal{F} = \sigma(S_0, S_1, ...S_N)$, where the $\{S_j \,|\, j = 0, ..., N\}$ are the basic assets traded in the market. We shall model the present worth of our position by $X \in \mathcal{L}_2(P)$, that is, essentially all random variables with finite variance. This somewhat restrictive framework greatly simplifies the proofs.

Definition 2.1 *A coherent risk measure is defined to be a function $\rho : \mathcal{L}_2 \to \mathbb{R}$ that satisfies the following axioms:*

1. *Translation Invariance: For any $X \in \mathcal{L}_2$ and $a \in \mathbb{R}$, we have $\rho(X + a) = \rho(X) - a$.*

2. *Positive homogeneity: For any* $X \in \mathcal{L}_2$ *and* $\lambda \geq 0$, *we have* $\rho(\lambda X) = \lambda\rho(X)$.

3. *Monotonicity: For any* X *and* $Y \in \mathcal{L}_2$, *such that* $X \leq Y$ *then* $\rho(X) \geq \rho(Y)$.

4. *Subadditivity: For any* X *and* $Y \in \mathcal{L}_2$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

These properties insure that diversification reduces the risk of the portfolio and if position size directly increases risk (consequences of lack of liquidity) then it is accounted in the future net worth of the position.

One example of coherent risk measures is the Conditional Value at Risk (CVaR). This measure indicates the expected loss incurred in the worst cases of the position. It is the most popular alternative to the Value at Risk measures.

$$CVaR_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha VaR_t(X)dt \ ,$$

where $VaR_t(X) = sup\{x : P[X > x] > t\}$

Spectral risk measures are defined by a general convex combination of Conditional Value at Risk.

Definition 2.2 *An element* $\phi \in \mathcal{L}_1([0,1])$ *is called an admissible risk spectrum if*

1. $\phi \geq 0$

2. ϕ *is decreasing*

3. $\|\phi\| = \int_0^1 |\phi(t)|dt = 1$.

Definition 2.3 *For an admissible risk spectrum* $\phi \in \mathcal{L}_1([0,1])$ *the risk measure*

$$\rho_\phi(X) = -\int_0^1 q_X(u)\phi(u)du$$

is called the spectral risk measure generated by ϕ.

ϕ is called the Risk Aversion Function and assigns, in fact, different weights to different p-confidence levels of the left tail. Any rational investor can express her subjective risk aversion by drawing a different profile for the weight function ϕ. Spectral risk measures are a subset of coherent risk measures as moved by

Acerbi (2002). Specifically, a spectral measure can be associated with a coherent risk measure that has two additional properties, law invariance and comonotone additivity. Law invariance in particular is a important property for applications since it is a necessary property for a risk measure to be estimable from empirical data.

Theorem 2.1 *The risk measure $\rho_\phi(X)$ be defined by*

$$\rho_\phi(X) = -\int_0^1 q_X(u)\phi(u)du \tag{1}$$

is a coherent risk measure. Here, for $u \in (0,1)$, $q(u) = \inf\{x \mid F(x) \geq u\}$ is the left continuous inverse of $F(x) = P(X \leq x)$.

Comment 2.1 *Note that if $X \geq 0$, then $q(u) \geq 0$ and $\rho(X) < 0$, that is, positive worth entails no risk.*

Example 2.1 *Conditional Value at Risk is a spectral risk measure defined by the Risk Aversion Function:*

$$\phi(p) = \frac{1}{\alpha} 1_{\{0 \geq p \geq \alpha\}}. \tag{2}$$

Example 2.2 *Another example of Risk Aversion Function is defined by Cotter and Dowd (2006)*

$$\phi(u) = \frac{Re^{-R(1-u)}}{1 - e^{-R}},$$

where R is the user's coefficient of absolute risk aversion.

Value at Risk is not a spectral risk measure because it is not a coherent risk measure and it does not satisfy the comonotone additive property.

On the other hand, Wang (1996) defines a family of risk measures by the concept of distortion function as introduced in Yaari's dual theory of choice under risk. Distortion risk measures are defined by a distortion function.

Definition 2.4 *We shall say that $g : [0,1] \to [0,1]$ is a distortion function if*

1. *$g(0) = 0$ and $g(1) = 1$.*

2. *g is non-decreasing function.*

For applications to insurance risk pricing it is convenient to think of the liabilities as positive variables, we restrict ourselves to $X \in \mathcal{L}_2^+(P)$, i.e., to positive random variables with finite variance, which we think about as losses or liabilities. If we were to relate this to the previous interpretation, we would say that our position is $-X$. The companion theorem characterizing the distorted risk measure induced by g is the following.

Theorem 2.2 *Define the distorted risk measure $D_g(X)$ induced by g on the class $\mathcal{L}_2(P)$ by*

$$D_g(X) = \int_0^\infty g(S(x))dx + \int_{-\infty}^0 [g(S(x)) - 1]dx, \tag{3}$$

where $S(x) = 1 - F_X(x)$. Then $D_g(X)$ has the following properties:

1. *$X \leq Y$ implies $D_g(X) \leq D_g(Y)$.*
2. *$D_g(\lambda X) = \lambda D_g(X)$ for all positive λ. $D_g(c) = c$ whenever c is a constant risk.*
3. *If the risks X and Y are comonotone, then $D_g(X+Y) = D_g(X) + D_g(Y)$.*
4. *If g is concave then $D_g(X+Y) \leq D_g(X) + D_g(Y)$.*
5. *If g is convex then $D_g(X+Y) \leq D_g(X) + D_g(Y)$.*

Hardy and Wirch (2001) have shown that a risk measure based on a distortion function is coherent if and only if the distortion function is concave. It can be shown that if g is concave the generated risk measure is spectral.

A distortion risk measure is the expectation of a new variable, with changed probabilities, re-weighting the initial distribution.

Example 2.3 *The VaR can be defined by the distortion function:*

$$g(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}, \tag{4}$$

Example 2.4 *CVaR is a distortion risk measure with respect to the following distortion function:*

$$g(x) = \begin{cases} \frac{x}{\alpha} & \text{if } x \leq \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}. \tag{5}$$

Example 2.5 *Some distortion risk functions used for insurance risk pricing are the following.*

1. *Dual-power functions:* $g(u) = 1 - (1-u)^{\nu}$ *with* $\nu \geq 1$.
2. *Proportional hazard transform:* $g(u) = u^{\frac{1}{\gamma}}$ *with* $\gamma \geq 1$.
3. *Wang's distortion function:* $g_{\alpha}(u) = \Phi[\Phi^{-1}(u) + \alpha], \quad u \in (0,1)$ *where* Φ *is the standard Normal distribution.*

More examples are the quadratic function or Denneberg's absolute deviation principle (see Wang (1996) for more details).

Wang's distortion function is often used to price financial and insurance risks (Wang (2002)). Wang's transform risk measure uses the whole distribution and hence accounts for extreme low-frequency and high severity losses.

Let us now recall a couple of results about quantiles. The following are taken from the nice expose by Laurent (2003). First of all we need the notion of quantile set.

Definition 2.5 *Given a probability space* $(\Omega, \mathcal{F}, P)$*, as above, a random variable* X*, and* $\alpha \in (0,1)$*, the* α*-quantile set of* X *is defined to be*

$$Q_X(\alpha) = \{x \in \mathbb{R} \,|\, P(X < x) \leq \alpha \leq P(X \leq x)\}.$$

Theorem 2.3 *With the notations introduced above*

$$Q_X(\alpha) = [q_X(\alpha), q_X^{+}(\alpha)],$$

where, as above

$$q_X(\alpha) = \inf\{x \,|\, P(X \leq x) \geq \alpha\} = \sup\{x \,|\, P(X < x) < \alpha\},$$

and

$$q_X^{+}(\alpha) = \sup\{x \,|\, P(X < x) \geq \alpha\} = \inf\{x \,|\, P(X \leq x) > \alpha\}.$$

The following characterizations are important: for $u \in (0,1)$ and $x \in \mathbb{R}$. We have:

$$\begin{array}{l} q_X^+(u) \geq x \Leftrightarrow P(X < x) \geq u \\ q_X(u) \leq x \Leftrightarrow P(X \leq x) \geq u. \end{array} \tag{6}$$

Also, for $u \in (0,1)$, the fact that $Q_X(u) = [q_X(u), q_X^+(u)]$, can be used to establish that

$$Q_{-X}(u) = -Q_X(1-u),$$

and in particular that $q_{-X}(u) = -q_X(1-u)$.

For the proof of the first theorem of section 3, we shall need the following version of the transference theorem (see section 6.5 in Kingman and Taylor(1966)). Set $G(x) = P(X < x) = F(x-)$. Then clearly $G(x)$ is increasing and left continuous. We have

Theorem 2.4 *(Transference theorem)*
(a) For every positive, measurable $h : (0,1) \to \mathbb{R}$ we have

$$\int_0^1 h(u)dq^+(u) = \int_{\mathbb{R}} h(G(x))dx,$$

where q^+ denotes the right quantile of $F(x) = P(X \leq x)$.
(b) For every positive, measurable $h : \mathbb{R} \to \mathbb{R}$, and every continuous increasing $g : [0,1] \to [0,1]$, we have

$$\int_0^1 h(q_X(u))dg(u) = \int_{-\infty}^{\infty} h(x)d(g \circ F_X)(x).$$

Proof: To prove (a) it suffices to prove the result for $h(u) = I_{(a,b]}(u)$ with $0 < a < b \leq 1$. In this case, involving the characterization (6), we have that

$$\int_0^1 I_{(a,b]}(u)dq^+(u) = q^+(b) - q^+(a) = \int_{\mathbb{R}} I_{(q^+(a),q^+(b)]}(x)dx = \int_{\mathbb{R}} I_{(a,b]}(G(x))dx,$$

which concludes the proof of (a). To prove (b), we consider $q_X : ((0,1), dg) \to \mathbb{R}$, and to identify the transfered measure it suffices to consider $h(x) = I_{(a,b]}(x)$. Denoting by $\tilde{g}$ the transfered measure, we have

$$\int_0^1 h(q_X(u))dg(u) = \int_{-\infty}^{\infty} h(x)d\tilde{g}(x),$$

therefore $\tilde{g}(b) - \tilde{g}(a) = \int_0^1 I_{(a,b]}(x)(q_X(u))dg(u) = \int_0^1 I_{(F_X(a),F_X(b)]}(u)dg(u)$
$= (g \circ F_X)(b) - (g \circ F_X)(a)$, for which we invoke (6) once more. $\square$

Corollary 2.1 *Under the assumptions of the theorem we have*

$$\int_0^1 h(u)dq^+(u) = \int_{\mathbb{R}} h(F(x))dx.$$

Proof: Just recall that G(x) differs from $F(x)$ at a countable set of points. □

3 Equivalence between spectral and distortion risk measures

In this section, we prove the relationship between spectral measures of risk and distorted measures.

Theorem 3.1 *Let ϕ be a piecewise continuous, admissible spectral function and let X be such that the spectral risk measure $\rho_\phi(-X)$ is finite. Then $D_g(X) \equiv \rho_\phi(-X)$ is a coherent distortion risk measure with concave distortion function satisfying $g'(u) = \phi(u)$*

Comment 3.1 *When integrating $g'(u) = \phi(u)$, keep in mind that $g(0) = 0$.*

Proof: Consider $\rho_\phi(-X) = -\int_0^1 q_{-X}(u)\phi(u)du$, and invoke part (b) of the transference theorem, with $h(x) = x$ and $\phi = g'$ to obtain

$$\rho_\phi(-X) = -\int_0^1 q_{-X}(u)\phi(u)du = -\int_{-\infty}^{\infty} x d\tilde{g}(x),$$

where $\tilde{g}(x) = g \circ F_{-X}(x)$. Consider now the following chain of identities.

$$\begin{aligned}
\int_{-\infty}^{\infty} x d\tilde{g}(x) &= \int_{-\infty}^{0} x d\tilde{g}(x) + \int_0^{\infty} x d\tilde{g}(x) \\
&= -\int_{-\infty}^{0} d\tilde{g}(x) \int_x^0 ds + \int_0^{\infty} d\tilde{g}(x) \int_0^x ds \\
&= -\int_{-\infty}^{0} ds \int_{(-\infty,s]} d\tilde{g}(x) + \int_0^{\infty} ds \int_{(s,\infty)} d\tilde{g}(x) \\
&= -\int_{-\infty}^{0} \tilde{g}(s)ds + \int_0^{\infty}(1 - \tilde{g}(s))ds.
\end{aligned}$$

Now, recalling that $\tilde{g}(s) = g \circ F_{-X}(x)$, after a simple change of variables, we can rewrite the last line of the previous chain as

$$-\int_{-\infty}^{0} (g \circ S_X)(s)ds + \int_0^{\infty} (1 - g \circ S_X)(s))ds = -D_g(X),$$

where, recall, $S_X(x) = P(X \geq x)$, thus concluding the proof. □

Example 3.1 *The risk measure CVaR is a spectral risk measure (see Example 2.1). If we apply the previous theorem to (2) we have that the Conditional Value at Risk is a distortion risk measure defined by:*

$$g(u) = \int_0^u \phi(s)ds = \int_0^u \frac{1}{\alpha} 1_{\{0 \geq s \geq \alpha\}} = \begin{cases} \frac{u}{\alpha} & \text{if } u \leq \alpha \\ 1 & \text{if } u \geq \alpha \end{cases}.$$

We thus reobtain the result of Example 2.4

The previous theorem admits the following reciprocal, the proof of which follows reversing the steps of the proof of the previous theorem.

Theorem 3.2 *Let g a concave distortion function, and let D_g be the associated distorted risk measure. Then $\phi(u) = g'(u)$ defines a spectral measure ρ_ϕ such that $\rho(X) = D_g(-X)$.*

Example 3.2 *We now calculate the Risk Aversion Function for the distortion risk functions listed in Example 2.5.*

1. *Dual-power measure: $\phi(u) = \nu(1-u)^{\nu-1}$ with $\nu \geq 1$.*
2. *Proportional hazard measure: $\phi(u) = \frac{1}{\gamma} u^{\frac{1}{\gamma}-1}$, with $\gamma \geq 1$.*
3. *Wang's measure: $\phi_\alpha(u) = e^{[-\alpha\Phi^{-1}(u) - \frac{\alpha^2}{2}]}$.*

Observe that for Proportional Hazard and Wang's measures, the Risk Aversion function is not bounded at zero. Moreover, the Risk Aversion function the Wang's measure decreases more quickly than that of Proportional Hazard's. Therefore, the investor using Wang's risk measure is more risk averse than an investor that measures the risk by the Proportional Hazard distortion because the first investor gives more importance to higher loses than the latter.

We have thus established that both methods to construct risk measures, either by means of distortion risk functions or by admissible spectral functions, are equivalent. In both, the risk measure can be thought of as a re-weighting of the initial distribution. Moreover, the derivative of the distortion risk function indicates the way of this re-weighting, as Balbas et. al. (2006) have indicated.

Comment 3.2 *These correspondences also provide an indirect proof of the fact that for a concave distortion function g, the risk measure defined by 3 is a coherent risk measure.*

4 Relationship to maximum entropy risk measures

Consider a random variable X describing the future value of some asset or market position. Assume that F_X is continuous and strictly increasing on its support. Another method to transform a given distribution appears when solving generalized moment problems by means of the relative entropy optimization. Denote by F_X^* the minimizer of the Kullback relative entropy $K(G,F) = \int \big(\ln dG/dF_X(x)\big)dG(x)$, over the class of probability distributions

$$\{G \,|\, \int h_i(x)dG(x) = \mu_i \;\; i = 1, ..., N\}.$$

It is well known, e.g., Kapur (1989), that there exists $(\lambda_1, ..., \lambda_N)$ such that

$$dF_X^*(x) = \exp(\sum_1^N \lambda_i h_i(x) - \Psi(\lambda))dF_X(x),$$

where

$$\Psi(\lambda) = \int e^{\sum_1^N \lambda_i h_i(x)} dF_X(x),$$

and $\lambda = (\lambda_1, \ldots, \lambda_N)$ is uniquely determined by the moment constraints.

The following is a formalization of an idea implicit in Reesor and McLeish's (2002) work. We begin exploring the nature of the distorted distribution function $F_X^*(x) = g(F_X(x))$. One such study was undertaken by Hurlimann in several papers, but it goes in a different direction than the one we follow here. We fit the results of Reesor and McLeish within this point of view, and establish a link between the risk measures defined by a relative entropy and a distortion risk measure, therefore a relationship between the spectral risk measures and the relative entropy risk measures.

We begin with a result in measure theory.

Theorem 4.1 *Let $dm^* = dF^*$ and $dm = dF$ be two measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $m^* << m$ having continuous density ψ. Then there exists a distortion function g such that $F^*(x) = g(F(x))$.*

Proof: Define $g(u) = \int_0^u \psi(q(s))ds$, where for $0 < u < 1$. We denote by $q(u)$ the left continuous inverse of F. Clearly, g is increasing, continuous, with $g(0) = 0$ and $g(1) = 1$.

Let us now verify that $g(F(x)) = F^*(x)$. An application of the transference theorem (or a variation on the change of variables theme, see Kingman and Taylor, (1970)) yields that

$$F^*(x) = \int_{-\infty}^x \psi(t)dF(t) = \int_{\mathbb{R}} I_{(-\infty,x])}(t)\psi(t)dF(t)$$

$$= \int_0^1 I_{(-\infty,x])}(F^{-1}(u))\psi(F^{-1}(u))du = \int_0^{F(x)} \psi(F^{-1}(u))du = g(F(x)),$$

since $u \leq F(x) \Leftrightarrow F^{-1}(u) \leq x$. $\square$

Comment 4.1 *This theorem asserts that any two possible distributions assigned to a given random variable, can be related by means of a distortion function.*

Theorem 4.2 *Let g be a piecewise continuously differentiable distortion function as above, and $F_X(x)$ be a continuous, and strictly increasing distribution function on its support. Then the measure $dm^* = dF_X^*$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, induced by the distorted distribution function F_X^*, is absolutely continuous with respect to $dm = dF_X$ having density $\psi(x) = g'(F_X(x))$.*

Proof: It boils down to noticing that $F_X^*(x) = g(F_X(x))$ implies that $dF_X^*(x) = g'(F_X(x))dF_X(x)$. $\square$

Comment 4.2 *In this case the relationship between distorted measures and spectral risk measures is easy to establish. Note that*

$$\begin{aligned} E^*[X] &= -E^*[-X] = -\int_{-\infty}^{\infty} x dF_{-X}^*(x) = -\int_{-\infty}^{\infty} x g'(F_{-X}(x))dF_{-X}(x) \\ &= -\int_0^1 q_{-X}(u)g'(u)du = \rho_\phi(-X) \end{aligned}$$

if we identify $\phi(u)$ with $g'(u)$.

The following result shows the relationship between spectral risk measures, and those based on distortion functions and relative entropy.

Theorem 4.3 *Let X be a risk such that F_X is continuous and strictly increasing on its support. Then the spectral, the distorted and the relative entropy risk*

measures are similar ways to transform the original distribution F_X. Denote by F^* the minimizer of the Kullback relative entropy described above, and assume as well that the $h_i's$ are continuously differentiable and that $\sum_{i=1}^N \lambda_i h_i'(u) \leq 0,\ \forall\ u \in [0,1]$, then there exists a concave distortion function, g and ϕ a admissible spectral function such that $E_{F^*}(X) = D_g(X) = \rho_\phi(-X)$ where F^* is the probability distribution obtained by the relative entropy minimization procedure.

Proof: If $dm^* = dF^*$ is the survival a relative entropy density, it is absolutely continuous with respect to the original distribution, $dm = dF$. By Theorem 4.1, there exists a distortion function, $g(u) = \int_0^u \exp\left(\sum_{i=1}^N \lambda_i h_i(q(s)) - \Psi(\lambda)\right) ds$, such that $g(F(x)) = F^*(x)$. We already seen that $E_{P^*}(X) = D_g(X)$. The distortion function is concave since

$$g''(u) = \exp\left(\sum_{i=1}^N \lambda_i h_i(u) - \psi(\lambda)\right) \sum_{i=1}^N \lambda_i h_i'(u)$$

and $\sum_{i=1}^N \lambda_i h_i'(u) \leq 0$. $\square$

We add the following simple observation: If $\int h_i(x) dF(x) = \mu_i$ are known generalized moments, and g is as in Theorem 4.1, then $F^*(x) = g(F(x))$ has moments $\int \hat{h}_i(x) dF^*(x) = \mu_i$ with $\hat{h}_i(x) = h_i(x)/g'(F(x))$.

Comment 4.3 *Consider two agents that assign different physical measures to their market models. Let $F^*(x)$ and $F(x)$ be the distribution function describing the statistical nature of some asset to each agent. Intuitively we may expect that $dF^* \sim dF$. What Theorem 4.1 asserts that upon some conditions on the density of F^* with respect to F, each may conclude that the other has a distorted view of reality with respect to him/herself.*

Comment 4.4 *Two agents may have the same point of view of reality, that is, both agents have the same market model, but may have different risk aversion functions, for example the first agent measures his level of risk by the distortion function g_1 and the agent two by g_2. If the agents have the same opinion about which losses are important, that is, the loss which they assign a new positive probability, or in the same sense the percentiles that they consider to measure the level of their risk.*

In this case, $F_1^*(x) = g_1(x)$ and $F_2^*(x) = g_2(x)$ are absolutely continuous one with respect to the other. Applying Theorem 4.1 we have $F_1^*(x) = h(F_2^*(x))$. If both

distortion functions are strictly increasing and continuous, the difference of the agent's risk aversion is given by $h = g_1 \circ g_2^{-1}$.

5 Conclusions

We have established that spectral risk measures are related to distorted risk pricing measures. Thus we have two representations at hand for a given risk measure, and may choose whichever representation is more convenient for the application at hand. Also, distorted risk pricing measures turn out to be absolutely continuous with respect to the measure that they distort. This allows us, for example, to interpret different physical probabilities (or different generalized scenarios) as distorted views of reality, one with respect to the other. Moreover, we link spectral risk measures, and those based on distortion functions and relative entropy.

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