Local Public Goods with Weighted Link Formation

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Abstract

We introduce weighted links in a local public good game in an endogenous network with heterogeneous players. We find that the equilibrium predictions are sharper than when links are not weighted. In particular, active players form a complete core-periphery graph, where they are either in the core of interconnected players, or connected to every player in the core. Furthermore, a player's type is tightly related to her public good provision and her position in the network.

Keywords: Weighted networks; network formation; public goods; free riding.
JEL classification: C72, D00, D85, H41.

1. Introduction

Inspired by influential contributions on global public goods (Warr, 1983; Bergstrom, Blume and Varian, 1986), some researchers have studied local public goods, which benefit only some individuals (Bramoullé and Kranton, 2007; Allouch, 2015, 2017). Indeed, individuals often choose between alternatives whose advantages they do not know. In order to take a decision, they acquire some information on these benefits either personally or through their peers. Since agents benefit from their neighbors' investment, personal acquisition of information is a local public good.

Most models study public good provision in fixed networks. However, for some applications, such as collecting and sharing information with friends, links are often endogenous as individuals decide with whom to interact in order to collect information. In a seminal contribution, Galeotti and Goyal (2010) allow homogeneous players to establish unweighted links, and show that strict Nash equilibria are core-periphery networks, in which large contributors are linked, while others link to them.

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Yet, individual characteristics are relevant in most applications. Consider consumers who decide how much information about alternative products to acquire (Feick and Price, 1987) or farmers who learn about new fertilizers (Conley and Udry, 2010). In these examples, richer consumers and farmers with bigger plots value more the same piece of information. Hence, these individual characteristics affect public good provision and networking; e.g., influential consumers (market mavens) enjoy shopping more (Feick and Price, 1987). This endogeneity creates a challenge for the estimation of the impact of social networks on behavior (Jackson, 2010).

Allowing for heterogeneity, however, significantly complicates the theoretical analysis. Kinateder and Merlino (2017, 2021) show that, when the benefits from public good consumption vary across players, they form nested split graphs, where players have nested neighborhoods.[1] However, the players who would demand more public good in isolation do not necessarily provide more. Hence, it is not possible to establish a tight relationship between a player's type, her position in the network and her contribution to the public good. This makes it difficult to employ these models in theoretical and empirical applications.

The contribution of this paper is to show that the equilibrium predictions are sharper if we allow players to establish weighted links. In the model, players simultaneously choose public good provision and weighted links. Links are established unilaterally, but once two players are linked, they access each other's public good provision proportionally to the weight of the link connecting them. Players differ in the (concave) valuation of consuming the public good; we then define better types as those who optimally acquire more public good in isolation.

We show that allowing players to establish weighted links implies that a player is active, i.e., contributing to the public good, only if she has completely exhausted the free riding opportunities, that is, only if there are no contributors left to whom she can profitably link. Indeed, when links are either 0 or 1, players who need a small additional amount of public good might prefer not to establish a link to a large contributor. When links are weighted, players can always modulate the weight of a link to their desire. Hence, a player contributes to the public good only after she has links of weight 1 to all large contributors.

We then characterize sociable Nash equilibria (Kinateder and Merlino, 2021) in which players establish all weakly profitable connections.[2] In that case, active players form a complete core-periphery graph, i.e., they are either in a core of fully-interconnected players, or in the periphery, in which case they establish a link of weight 1 to all players in the core. This result stems from the fact that, as we have just discussed, active players establish links of weight 1, together with two other

[1]When homogeneous players' efforts are strategic complements, nested split graphs also emerge if the value function is convex (Hiller, 2017).

[2]In other words, a Nash equilibrium is sociable if no player can increase the weight of a link (without changing other links), while obtaining the same payoffs. This refinement prevents situations in which, out of indifference, many networks can be an equilibrium.

equilibrium requirements: first, conditional on linking, a player links to the largest contributors; and second, the largest contributors form a core, as otherwise they could increase their utility by linking among them. Hence, while weighted links discipline the behavior of active players, none of them establishes any weighted link in equilibrium.

In any non-empty sociable equilibrium network, only inactive players, i.e., those who do not contribute to the public good, establish weighted links. Additionally, there are no isolated players, as a weighted link to the largest contributor is profitable also for players with very low valuations.

We then derive the implications of the equilibrium characterization for players' public good provision and their linking behavior. First, only players of the best type can be in the core, and the size of the core is bounded by the cost of establishing links. Second, better types have more links and contribute more to the public good, so that players are inactive only when their type is sufficiently low.

Hence, in contrast to models which restrict links to be either 0 or 1, in our model there is a tight relationship between players' type, their public good provision and their network position. This implies that a player's type can be inferred from her network position or her public good provision.

Finally, we discuss the robustness of the equilibrium characterization of the model in Section 4. There, we characterize strict Nash equilibrium networks and discuss the role of alternative assumptions, such as different linking technologies, one-way flow of spillovers, or the introduction of a fixed cost of linking.

This paper naturally relates to Galeotti and Goyal (2010). They assume that players are homogeneous and links can be either 0 or 1; then, strict Nash equilibrium networks are complete core-periphery graphs. Allowing for heterogeneous players, Kinateder and Merlino (2017) show how the characterization is affected by the source of heterogeneity. Kinateder and Merlino (2021) study income redistribution and inequality in a more general model with non-linear best replies and a budget constraint. The contribution of this paper is instead to allow for weighted links.

There are a few papers that study the possibility of establishing weighted links. In particular, Bloch and Dutta (2009) and Baumann (2021) develop network formation models with weighted links, but without strategic interaction on the resulting network. Hence, differently from here, in those models the value of connections does not depend on players' level of activity.[3]

We model network formation non-cooperatively (Bala and Goyal, 2000). When players only choose links, heterogeneity in valuation plays a minor role (Galeotti, Goyal and Kamphorst, 2006). Yet, it significantly affects the equilibria when linking costs are also heterogeneous (Billand, Bravard and Sarangi, 2011) to a point that Nash networks might not exist (Haller, Kamphorst and Sarangi,

[3]Cabrales, Calvó-Armengol and Zenou (2011), Galeotti and Merlino (2014) and Merlino (2014, 2019) introduce strategic interaction when players decide how much to invest in an undifferentiated socialization effort, so that players decide how much to link, but not with whom.

2007). In our model, the players' benefit from linking is determined endogenously by their choice of effort.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the main results, while Section 4 discusses their robustness. Section 5 concludes. All proofs are in the appendix.

2. Model

We study a local public good game, in which each player $i \in N = \{1, ..., n\}$ exerts effort $x_i \in X$, where $X = [0, +\infty)$ and establishes costly weighted links to free ride on the effort exerted by others.

Denote a weighted directed network g by an adjacency matrix in which each line i represents player i's links by a row vector $g_i = (g_{i1}, ..., g_{in})$, where $g_{ij} \in [0, 1]$, for each $j \in N \setminus \{i\}$ and $g_{ii} = 0$ for all $i \in N$. Let $g_i \in G_i = [0, 1]^{n-1}$. Network formation is non-cooperative (Bala and Goyal, 2000): player i links to j with weight g_{ij} at a cost $k \cdot g_{ij} > 0$. We show below that direct spillovers are never negative, so that incoming links are always accepted. We say that player i links to player j if $g_{ij} > 0$. We define $N_i(g) = \{j \in N : g_{ij} > 0\}$ as the set of players to whom i links, i.e., the set of i's neighbors.

Spillovers flow among connected players, independently of who initiated the link. This model thus captures situations where one player pays for the communication or link, but then information is exchanged between both. Examples for this are the acquisition and exchange of information about new products and technologies with friends and social acquaintances (Feick and Price, 1987; Conley and Udry, 2010). More formally, we represent the network of spillovers by $\bar{g}$, the closure of g. In the undirected network $\bar{g}$, $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$ for each $i, j \in N$; $N_i(\bar{g}) = \{j \in N : \bar{g}_{ij} > 0\}$ is the set of players i is linked to in $\bar{g}$, and let $\eta_i(\bar{g}) = |N_i(\bar{g})|$ be the number of i's neighbors in $\bar{g}$, or i's *degree*. A player i is isolated if $\bar{g}_{ij} = 0$ for all $j \in N \setminus \{i\}$.

A *core-periphery graph* is a network such that for every pair of players l, m in the core $\mathcal{C}(\bar{g})$, $\bar{g}_{lm} = 1$, while for every pair of players i, j in the periphery $\mathcal{P}(\bar{g})$, $\bar{g}_{ij} = 0$; furthermore, for any $i \in \mathcal{P}(\bar{g})$, there exists $l \in \mathcal{C}(\bar{g})$ such that $\bar{g}_{il} = 1$.[4] A *complete core-periphery graph* is a core-periphery network where $\bar{g}_{il} = 1$ for all all $i \in \mathcal{P}(\bar{g})$ and $l \in \mathcal{C}(\bar{g})$. A *weighted core-periphery graph* is a network such that for every pair of players l, m in the core $\mathcal{C}(\bar{g})$, $\bar{g}_{lm} > 0$, while for every pair of players i, j in the periphery $\mathcal{P}(\bar{g})$, $\bar{g}_{ij} = 0$; furthermore, for any $i \in \mathcal{P}(\bar{g})$, there exists $l \in \mathcal{C}(\bar{g})$ such that $\bar{g}_{il} > 0$. A *star* is a (weighted) core-periphery network with a single player in the core.

Let $\lfloor y \rfloor$ be the floor operator of any $y \in \mathcal{R}^+$. Following Mahadev and Peled (1995), in a *nested split graph $\bar{g}$ with degree partition $D = (D_0, D_1, ..., D_K)$*,[5] nodes can be partitioned in independent

[4]Differently from Galeotti and Goyal (2010), we use the standard definition of core-periphery graphs, whereby players in the core can have different neighborhoods.

[5]Take $\bar{g}$ whose positive degrees are $d_{(1)} < d_{(2)} < ... < d_{(K)}$ and let $d_0 = 0$ (even if there is no agent with degree 0

sets D_i for $i = 1, ..., \lfloor \frac{K}{2} \rfloor$ containing the periphery players and a dominating set of core players $\cup_{i=\lfloor \frac{K}{2} \rfloor+1}^{K} D_i$ in the connected subgraph of $\bar{g}$ constituted by $N \setminus D_0$. Moreover, the neighborhoods of the nodes are nested. More formally, for each node $v \in D_i$, for $i = 1, ..., K$:

$$N_v(\bar{g}) = \begin{cases} \cup_{j=1}^{i} D_{K+1-j} & \text{if } i = 1, ..., \lfloor \frac{K}{2} \rfloor, \\ \cup_{j=1}^{i} D_{K+1-j} \setminus \{v\} & \text{if } i = \lfloor \frac{K}{2} \rfloor + 1, ...K. \end{cases}$$

As players in the core are all connected among themselves and all non-isolated players link to them, a nested split graph is a core-periphery graph. As the set of agents' neighbors are nested, for any pair of agents $i, j \in \mathcal{P}(\bar{g})$, if $\eta_j(g) \leq \eta_i(g)$, then $N_j(g) \subseteq N_i(g)$. In the same spirit, we define a *weighted nested split graph* as a weighted core-periphery graph such that, for any $i, j \in \mathcal{C}(\bar{g})$, if $\sum_{z \in N} \bar{g}_{jz} < \sum_{z \in N} \bar{g}_{iz}$, then $\bar{g}_{jz} \leq \bar{g}_{iz}$ for all $z \in \mathcal{C}(\bar{g})$ and, furthermore, for any $i, j \in \mathcal{P}(\bar{g})$, if $\sum_{z \in N} \bar{g}_{jz} \leq \sum_{z \in N} \bar{g}_{iz}$, then $\bar{g}_{jz} \leq \bar{g}_{iz}$ for all $z \in \mathcal{C}(\bar{g})$.

Player i's set of strategies is $S_i = X \times G_i$, and $S = S_1 \times ... \times S_n$. Denote a strategy profile by $s = (x, g) \in S$. Player i's payoffs are:

$$U_i(x, g) = f_i\Big(x_i + \sum_{j \in N} \bar{g}_{ij} x_j\Big) - c x_i - \sum_{j \in N} k g_{ij}, \tag{1}$$

where $c > 0$ is the cost of providing the public good, $k > 0$ is the linking cost and $f_i(x)$ describes player i's benefits from public good consumption. Furthermore, for all $i \in N$, *(i)* $f_i(x)$ is a twice continuously differentiable function in x and i, and it is strictly concave and increasing in x, *(ii)* $f_i'(0) > c$, *(iii)* $lim_{x \to \infty} f_i'(x) = m_i < c$, and *(iv)* $\partial^2 f_i / \partial x \partial i \leq 0$, or $f_i'(x) \geq f_j'(x)$ for all $x > 0$, if $i < j$. Hence, there is a unique positive optimal investment in the public good in isolation for every i denoted by $a_i = \arg\max_{x_i \in X} f_i(x_i) - c x_i$, which represents player i's type. We assume $a_1 \geq a_2 \geq ... \geq a_n > 0$. Hence, we refer to lower-indexed players as *better types*. We say that a player i is *active* if $x_i > 0$, and *inactive* otherwise.

A strategy profile $s^* = (x^*, g^*)$ is a *Nash equilibrium* if for all $s_i \in S_i$ and all $i \in N$, $U_i(s^*) \geq U_i(s_i, s_{-i}^*)$, where $s = (s_i, s_{-i})$. Following Kinateder and Merlino (2021), we say that a Nash equilibrium s^* is *sociable* if, whenever there exist $i \in N$ and $s_i' \neq s_i^*$ such that $U_i(s^*) = U_i(s_i', s_{-i}^*)$, then $g_{ij}^* \geq g_{ij}'$ for any $j \in N \setminus \{i\}$, with strict inequality for some j. In words, in a sociable equilibrium, any player who is indifferent between increasing the weight of one link or not, increases it; hence, if there is no such indifference, a Nash equilibrium is sociable. Finally, an equilibrium is *strict* if no player can unilaterally change her strategy without reducing her payoff. It follows that any strict equilibrium is sociable.

in $\bar{g}$). Further, define $D_j = \{i \in N : d_i(\bar{g}) = d_{(j)}\}$ for $j = 0, ..., K$. Then, the set-valued vector $D = (D_0, D_1, ..., D_k)$ is called the degree partition of $\bar{g}$.

3. Results

First, we show that, as in the model with unweighted links, in any equilibrium, active players always consume exactly the amount of public good as if they were in isolation.

Lemma 1 *In any Nash equilibrium (x^*, g^*), $x_i^* + \sum_{j \in N} \bar{g}_{ij} x_j^* = a_i$ for all $i \in N$ such that $x_i^* > 0$.*

The proof of this result adapts that of Lemma 2 in Kinateder and Merlino (2017) to our model with weighed links.

Note that, when players are homogeneous, the introduction of weighted links does not affect the equilibrium characterization of Galeotti and Goyal (2010). Indeed, when all players want to consume a_1 in isolation, every two players providing k/c or more public good are linked. Otherwise, they could (weakly) increase their payoffs by establishing that link, since the cost of free-riding is lower than that of own provision. Other players then link to all players in the core. As a result, in a strict Nash equilibrium the total provision of public good is a_1, a complete core-periphery graph emerges and players do not use weighted links. If instead the largest contributors provide exactly k/c and are not all linked among themselves, then the total provision may exceed a_1. In this case, players may use weighted links out of indifference, but this does not affect the equilibrium characterization.

To characterize equilibrium networks for heterogeneous players, define $\overline{k} = ca_1$.[6]

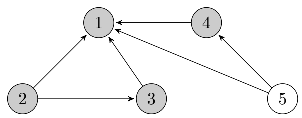
Theorem 1 *A sociable Nash equilibrium always exists. If $k \leq \overline{k}$, in any sociable Nash equilibrium (x^*, g^*):*
(i) $\bar{g}^*$ is a weighted core-periphery graph;
(ii) active players form a complete core-periphery graph with links of weight 1;
(iii) there are no isolated players.

In any sociable Nash equilibrium, the largest contributors form a core (Theorem 1.*(i)*). If there were several players providing k/c of the public good who are not linked among themselves, there could be Nash networks without a core. In contrast to the model with homogeneous players, in this model this also might happen when the total public good provided is a_1. To show this, Figure 1 depicts an equilibrium where some players provide k/c. While all of them link to the highest contributor (so that the total public good provision is a_1), the equilibrium network does not display a core.[7]

[6] If $k > \overline{k}$, the unique equilibrium network is empty.

[7] Perturbations of the values of the linking cost do not suffice to break eventual ties. For example, consider an economy with 5 players where $a_1 = 4$, $a_i = 3$ for the other players. The following is a Nash equilibrium without core for any $k \in [c/2, c]$: $x_1^* = 2$, $x_2^* = x_3^* = k/c$, $x_4^* = x_5^* = 1 - k/c$, $g_{i1}^* = 1$ for all $i \neq 1$, $g_{42}^* = 1$ and $g_{53}^* = 1$, while no other links are formed. Since 2 and 3 receive links but are not themselves connected, no core emerges for an interval of values of the linking cost.

In a sociable equilibrium, however, all players providing k/c or more are linked among themselves, ensuring the formation of the core.[8]

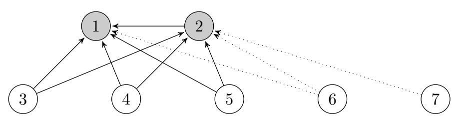


Player	1	2	3	4	5
a_i	1	.8		.6	
x_i^*	.4	.2			0

Figure 1: Example of a Nash equilibrium that is not a core-periphery graph when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .2$. Actives players are in gray.

Other players establish links to as many players in the core as it is profitable for them. As a result, a core-periphery graph emerges.

The main result of this theorem is Theorem 1.*(ii)*: active players establish only links of weight 1, so that in a sociable Nash equilibrium they form a complete core-periphery graph. In other words, they are either in the core, thereby having a link of weight 1 with all players in the core, or in the periphery and sponsor themselves a link of weight 1 to all the players in the core. The intuition for this result is the following. Since core players produce at least k/c, free-riding on them is cheaper than own provision. Hence, players are active only if they link to all core players, thereby exhausting all free riding opportunities. For example, in Figure 2, players 1 to 5 are active, and form a complete core-periphery.



Player	1 & 2	3,4,5	6	7
b_i	2	1.78885	1.4	.6
a_i	1	.8	.49	.09
x_i^*	.35	.1	0	0
links	$\bar{g}_{12} = 1$	$g_{i1} = g_{i2} = 1$	$g_{61} = .97, g_{62} = .9356$	$g_{71} = .35$

Figure 2: Example of a sociable Nash equilibrium with $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .3$. Core players are in gray and weighted links are dotted.

It is worthwhile to discuss some other novel features of this model. First, in general, players in the core might provide identical quantities of the public good, and thus receive links from different players who are indifferent to whom to link. As a result, periphery players might link to different players in the core and have several weighted links. Figure 2 depicts an example where players 1 and 2 provide identical quantities, so that players who link to only one of them, such as 7, are

[8] In the example of Figure 1, sociability would require players 4 and 5 to link either to player 2 or 3, resulting in a core-periphery graph.

indifferent to whom to link. Hence, non-nested neighborhoods might emerge. We show in Section 4 below that the predictions of our model are even sharper if we focus on strict Nash equilibria.

Second, there are no isolated players. The intuition is the following. Whenever the network is non-empty, someone, say player i, provides at least k/c. As then free-riding is cheaper than own provision, all players find it profitable to establish a link with the appropriate weight to player i. This contrasts with models of unweighted links, as there, if a player has a very low valuation of the public good, she might prefer staying isolated and active instead of establishing a full link to the largest contributor.

In the next proposition, we characterize how players' type is related to network position and the size of the core. For this purpose, let $|a_1|$ be the number of players of the best type.

Proposition 1 *In any non-empty sociable Nash equilibrium* (x^*, g^*)*:*
(i) $a_i = a_1$ *for all* $i \in \mathcal{C}(\bar{g}^*)$*;*
(ii) $|\mathcal{C}(\bar{g}^*)| \leq \min\{\lfloor ca_1/k \rfloor, |a_1|\}$*;*
(iii) $\sum_{i \in N} x_i^* = a_1$*.*

Since all players in the core consume the same amount of public good as they are connected to all active players, only players of the best type can be in the core. Indeed, if a player of a lower type were in the core, she would access at least as much public good as players of the best type, which by Lemma 1 cannot be an equilibrium. Hence, all players whose consumption in isolation is below a_1 are always in the periphery.

Proposition 1.*(i)* implies that players in the core collectively provide at most a_1. This allows us to bound the size of the core. Indeed, the maximal size is achieved when all core players provide the same amount of public good; in this case, each of them provides $a_1/|\mathcal{C}(\bar{g}^*)|$. However, in order to attract links, each player in the core has to provide at least k/c. These two facts yield the threshold stated in Proposition 1.*(ii)*.

Furthermore, as active agents form a complete core-periphery graph and there are no isolated players, total provision must then be equal to a_1 (Proposition 1.*(iii)*).

Building on Proposition 1, we can now derive some general results on how players' type relates to their provision and their neighborhood in $\bar{g}$.

Proposition 2 *In any non-empty sociable Nash equilibrium* (x^*, g^*) *if* $a_i > a_j$ *for* $i, j \in N$*, then:*
(i) $\sum_{z \in N} \bar{g}_{iz}^* \geq \sum_{z \in N} \bar{g}_{jz}^*$*, with strict inequality if* $x_i^* = x_j^* = 0$*;*
(ii) $x_i^* \geq x_j^*$*, with strict inequality if* $x_j^* > 0$*.*

In words, in the model with weighted links, there is a tight relationship between players' type, provision and links. First, better types have more links, and strictly so if they are inactive. The intuition is the following: since players of better type need more public good, they have to provide more, in which case they receive more links, and/or link more to free ride on others.

Furthermore, even when there are multiple equilibria, better types provide more public good. When links are discrete, better types in the periphery have more links as well, but low type players, who have a low valuation of the public good, might prefer not to link to all players in the core, and instead contribute a bit. However, in this model, active players in the periphery have the same spillovers, and inactive players of better type have more links. These results imply that better types both link and contribute more also in the periphery.

The next proposition further characterizes the relationship between players' types and their public good provision.

Proposition 3 *In any non-empty sociable Nash equilibrium* (x^*, g^*)*:*
(i) $x_i^* > x_j^*$ *for some* $i, j \in \mathcal{C}(\bar{g}^*)$ *if, and only if,* $\sum_{z \in N} \bar{g}_{iz}^* x_z^* < \sum_{z \in N} \bar{g}_{jz}^* x_z^*$*, while* $\sum_{z \in N} \bar{g}_{iz}^* \geq \sum_{z \in N} \bar{g}_{jz}^*$*;*
(ii) there can be $i \in N$ *such that* $a_i < a_1$ *and* $x_i^* > 0$ *only if, for all* j *with* $a_j = a_1$*,* $j \in \mathcal{C}(\bar{g}^*)$*;*
(iii) there exists a threshold $\tilde{n}$ *such that* $x_i^* = 0$ *for all* $i \geq \tilde{n}$*;*
(iv) if $x_i^* = 0$ *and* $a_i < a_1$*,* $\sum_{z \in N} g_{iz}^* x_z^* \geq a_i$*.*

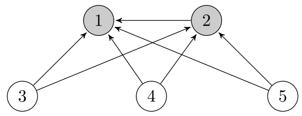
First, since all players in the core are of the best type, they consume the same amount of the public good. Hence, if one of them provides more, she has to receive less spillovers, as in Figure 3. Furthermore, since she provides more, she might attract more links, but only from inactive players.

However, while all players in the core are of type 1 (Proposition 1.*(i)*), not all players of type 1 need to be in the core. Whether any of them is in the periphery has implications for how much public good other periphery players provide. As Proposition 3.*(ii)* points out, players of a lower type—who are then in the periphery—can be active only when all players of type 1 are in the core.

To see why, consider the following cases. First, there can be at most one active player of the best type in the periphery, as otherwise active players of the same type would consume different amounts of the public good, thereby contradicting Lemma 1. In that case, all other periphery players must be inactive by the same reasoning. Figure 3 gives an example of such an equilibrium, where players 1 to 3 are of the best type, but player 3 is in the periphery and active. It is easy to see that if players 4 and 5 were active, player 3 would consume less than players 1 and 2, thereby leading to a contradiction.

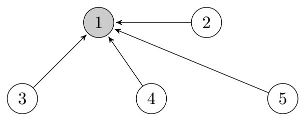
Additionally, if there is more than one player of the best type in the periphery, all of them must be inactive, otherwise they would consume different amounts of the public good. Figure 4 exhibits an example of such an equilibrium with a star, but the same logic applies to all other equilibrium networks, as active players have to be arranged in complete core-periphery graphs.

As spillovers crowd out own contribution, we can find a threshold that identifies players who are inactive in all equilibria, as the provision of players in the core always satisfies their demand for the public good (Proposition 3.*(iii)*).



Player	1	2	3	4,5
b_i		2		1.8
a_i		1		.81
x_i^*	.5	.4	.1	0
links	-	$g_{21} = 1$		$g_{i1} = g_{i2} = 1$

Figure 3: Example of a sociable Nash equilibrium with one best type in the periphery when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .3$. Core players are in gray.



Player	1	2,3	4,5
b_i		2	1.8
a_i		1	.81
x_i^*	1	0	0
links	-	$g_{i1} = 1$	$g_{i1} = 1$

Figure 4: Example of a sociable Nash equilibrium with more than one best type in the periphery when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .3$. Core players are in gray.

Additionally, inactive players generally consume more than in isolation, as stated in Proposition 3.(iv). Indeed, while the marginal cost of own provision is given by c, the marginal cost of linking is only k. Hence, in any non-empty network, the marginal unit of public good is cheaper for inactive players. For example, in the equilibrium depicted in Figure 2, players 6 and 7 consume .66 and .1225, respectively, which are both higher than their consumption in isolation.

To summarize, in this section we have derived the equilibrium characterization of a local public good model when players can establish weighted links. The starkest implication of allowing players to form weighted links is that active players are arranged in complete core-periphery structures where all links are of weight 1. This translates into a tight relationship between a player's type, her position in the network and her contribution to the public good.

In the following section, we discuss some additional properties and extensions to the benchmark model.

4. Discussion and Extensions

4.1. Strict Nash Equilibria

We now show that the characterization of strict Nash equilibria is sharper than that of sociable equilibria.[9]

Proposition 4 *In a strict Nash equilibrium* (x^*, g^*)*,* $\bar{g}^*$ *is a weighted nested split graph where active players have links of weight* 1 *and inactive players have at most one link with weight in* $(0, 1)$.

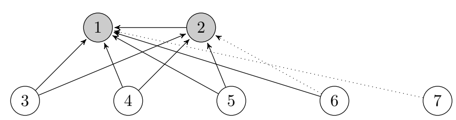
[9]Note that all strict Nash equilibria are sociable, but the converse is not true.

In a strict equilibrium, any player is neither indifferent between linking and public good provision, nor between linking to different players. Hence, it is best to always exhaust a link to a contributor before linking to another who provides less. As a result, weighted nested split graphs emerge.

Additionally, only the link to the lowest contributor among the out-links of an inactive player can be weighted. Hence, players have at most one weighted link.

Figure 5 depicts a strict equilibrium for the same economy as in Figure 2. In this example, since linking to player 1 is more profitable than linking to 2, every player establishes a link with a weakly higher weight to 1 than to 2. As a result, now neighborhoods are nested, and inactive players only have one weighted link.

This simple fact has some consequences for welfare, as inactive players have a different number of links. As each core player provides a different amount of the public good than in the equilibrium depicted in Figure 2, the weight of the links is different. In particular, player 6 has a weighted link to 2, and 7 to 1. Since player 1 is providing more, and 2 less, than in the equilibrium in Figure 2, now 6 links (and thereby consumes) less, and 7 more.



Player	1	2	3,4,5	6	7
b_i	2		1.78885	1.4	.6
a_i	1		.8	.49	.09
x_i^*	.37	.33	.1	0	0
links	-	$g_{21} = 1$	$g_{i1} = g_{i2} = 1$	$g_{61} = 1, g_{62} = .68$	$g_{71} = .37$

Figure 5: Example of a strict Nash equilibrium with $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .3$. Core players are in gray and weighted links are dotted.

4.2. Linking technology

In the benchmark model, we assume that the technology to establish links is linear, in the sense that the weight of a link between two players is equal to their investment in that link, up to the maximal weight of 1. In this subsection, we consider a different link formation technology such that only weighted links are established.

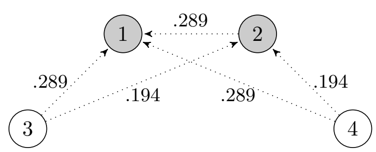
More formally, define by $e_{ij} \geq 0$ player i's investment in a link with player j. Then, $g_{ij} = l(e_{ij})$, where the linking technology $l(\cdot)$ is an increasing and differentiable function such that $l(0) = 0$, $\lim_{e\to\infty} l(e) = 1$ and $l(e) < 1$ for any $e \in [0, \infty)$. Let $e_i = (e_{i1}, ..., e_{in})$, where $e_{ij} \in [0, \infty)$, for each $j \in N \setminus \{i\}$ and $e_{ii} = 0$, for all $i \in N$. Let $e_i \in E_i = [0, \infty)^{n-1}$. Player i's set of strategies is then defined accordingly as $S_i = X \times E_i$.

Next we show that with this linking technology any non-empty sociable equilibrium yields a weighted core-periphery graph.[10]

Proposition 5 *If the linking technology is $l(\cdot)$, in a non-empty sociable Nash equilibrium (x^*, g^*), $\bar{g}^*$ is a weighted core-periphery graph, where $g^*_{iz} = g^*_{jz}$ for all $i, j, z \in N$ with $x^*_i, x^*_j \in (0, x^*_z)$.*

This proposition stresses that the characterization of the benchmark model is robust to different linking technologies. However, there are some important differences. Indeed, the weight of a link to players depends on their public good provision. So, when links are weighted and core players provide a different amount of the public good, they might receive different spillovers from periphery players. Hence, while players of a better type are still in more central positions in the network, players in the core can be of different type, as shown in the following example.

Figure 6 depicts an equilibrium of an economy where $l(e) = 1 - exp(-e)$. In the example, both players 1 and 2 receive links from players 3 and 4. However, the weights of the links are different, so that players 1 and 2 receive different spillovers. Therefore, both of them are in the core even if they have different public good consumption in isolation.



Player	1	2	3	4
b_i	2	1.9	1.5	1.4
a_i	1	.9025	.5625	.49
x^*_i	.703	.62	.239	.166

Figure 6: Example of a sociable Nash equilibrium with linking technology $l(e) = 1 - exp(-e)$ when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .5$. Core players are in gray and the weight of a link is indicated on the corresponding edge.

4.3. Fixed cost in linking

In the benchmark model, we have assumed that the costs to bear in order to form a link are directly proportional to the weight of that link. In some applications however, there might be fixed costs associated with establishing a link, such as starting a phone call. These concerns are captured in the following formulation:

$$\sum_{j \in N_i(g)} (K + kg_{ij}), \tag{2}$$

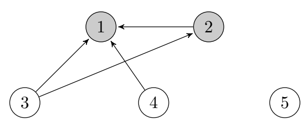
where K is the fixed component to be paid for each sponsored link independently of its weight. Note that this formulation embeds both the unweighted and the weighted link model for $k = 0$ and $K = 0$, respectively.

[10]If $k > \bar{k}$, the empty network is the unique equilibrium, where $\bar{k} = ca_1 \lim_{e \to 0} l'(e)$.

The characterization that results under (2) is the same as in a model of unweighted links (Theorem 2 in Kinateder and Merlino, 2017). While active agents still only have links of weight 1, the fixed cost of linking might induce players to provide a little bit of public good rather than establishing an additional link. The following corollary states these results.

Corollary 1 *For any $K > 0$, in any sociable Nash equilibrium (x^*, g^*), $\bar{g}^*$ is a weighted core-periphery graph where active players have links of weight 1. Furthermore, there exists an f_i such that player i is active and not connected to all players in the core $\mathcal{C}(\bar{g}^*)$.*

This corollary stresses that better types might not be larger contributors and active agents do not need to form a complete core-periphery graph; this second result also implies that there can be isolated players. The example depicted in Figure 7 exhibits these results.



Player	1	2	3	4	5
b_i	4.11825	4	4	3.1	1
a_i	4.24	4	4	2.4025	.25
x_i^*	2	1.6	.4	.4025	.25

Figure 7: Example of a sociable Nash equilibrium in the model with fixed costs of linking where all players are active but g^* is not a complete core-periphery graph when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$, $K = 1$ and $k = .5$. Core players are in gray.

Note, however, that the results of the model with $K = 0$ are robust to the introduction of a small fixed cost of linking. As we show in the following corollary, there exists a threshold of the fixed cost of linking below which active players behave as when $K = 0$, while inactive players link and consume less than when $K = 0$.

Corollary 2 *Consider the model where linking costs are given by (2). Take a strict Nash equilibrium (x^*, g^*) when $K = 0$. Then, there exists $\kappa > 0$ such that (x^*, g^*) is a strict Nash equilibrium for any $K < \kappa$.*

4.4. One-way flow of information

In some cases, information does not flow two-way, but to access it each player has to sponsor a link, as on Twitter. Our model extends to this case, in which sponsoring a link only allows its sponsor to access information.

Formally, the payoff function (1) is modified and becomes

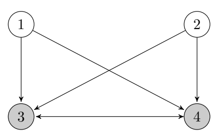
$$U_i(x, g) = f_i\Big(x_i + \sum_{j\in N} g_{ij}x_j\Big) - cx_i - \sum_{j\in N} g_{ij}k, \tag{3}$$

i.e., information does not any more flow on the closure of $\bar{g}$, but rather on g. Then, we derive the following:[11]

Corollary 3 *A sociable Nash equilibrium (x^*, g^*) with one-way flow of information always exists, and g^* is either empty or a weighted core-periphery graph where active agents have links of weight 1. Moreover, there is $\underline{n}$ such that $j \in \mathcal{P}(g^*)$ for all $j \geq \underline{n}$ in all equilibria (x^*, g^*).*

This result shows that the characterization of Theorem 1 holds. In the core, players reciprocate links, and periphery players link to core players. However, since periphery players in the one way-flow model do not create spillovers, they can achieve a higher consumption than a player in the core by complementing the spillovers they access with their own provision. As a result, contrarily to Proposition 1, the best types need not be in the core.

For example, in Figure 8, the players' demand for public good is sufficiently similar, and thus, having those of the best type in the periphery constitutes an equilibrium.



Player	1,2	3,4
a_i	1	.8
x_i^*	.2	.4

Figure 8: Example of a sociable Nash equilibrium in the model with one-way flow of spillovers where best types are in the periphery when $f_i(x_i) = b_i\sqrt{x_i}$, $c = 1$ and $k = .3$. Core players are in gray.

4.5. Two-sided link formation

The equilibrium characterization we derived so far assumes that players can form links unilaterally. This assumption does not capture well situations where some investment by both players is possible/required for a link between them to be established. In the following, we argue that our characterization is robust to this extension as long as the complementarity between the two players' investments is not too strong, or players can transfer resources to compensate for linking costs.[12]

Following Ding (2019), we can express the linking technology as

$$\bar{g}_{ij} = \mathbf{1}\left\{\left(\frac{1}{2}e_{ij} + \frac{1}{2}e_{ji}\right)^{\frac{1}{\beta}} \geq 1\right\}, \tag{4}$$

[11]In the one-way flow model, we adapt the definition of the core as follows: for every $i, j \in \mathcal{C}(g)$, $g_{ij} = g_{ji} = 1$. Hence, all links in the core are reciprocated.

[12]In case mutual consent is required to establish a link (Jackson and Wolinsky, 1996), players may ask for compensation to share the public good they provide (Bloch and Jackson, 2007). Then, each equilibrium network under one-sided linking is an equilibrium under two-sided link formation with transfers since the player proposing a link under one-sided linking can always find transfers so that the link is accepted by the other party under two-sided linking with transfers.

where **1** is the indicator function and e_{ij} and e_{ji} are i's and j's investment in an undirected link $\bar{g}_{ij}$ between them, for $i, j \in N$ and $e_{ij} \geq 0$. As in the benchmark model, the undirected network $\bar{g}$ resulting from a profile of investment e_{ij} for all $i, j \in N$ with $i \neq j$, is the network that describes spillovers across players in (1). The linking technology described in (4) corresponds to one-sided linking if $\beta \to \infty$.

Ding (2019) shows that with (4), only one player would sponsor a link as long as $\beta > 1$. It is then easy to see that a player links to j if $x_j \geq 2^{\frac{1}{\beta}} k/c$. Hence, any two players providing more than $2^{\frac{1}{\beta}} k/c$ are linked in equilibrium, and the same equilibrium characterization results for any $\beta > 1$.

5. Conclusion

In this paper, we find that allowing for weighted links in a local public good game with an endogenous network yields a tight relationship between type, public good provision and network position. Hence, players' types can be inferred from their network position or their public good provision, and vice versa. We thus believe, this model can be used to guide future empirical or theoretical work in applications when the network is endogenous and links are weighted.

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Appendix

Proof of Lemma 1. Suppose that (x^*, g^*) is a Nash equilibrium and $x_i^* > 0$. However, suppose *ad absurdum* that $x_i^* + \sum_{j \in N} \bar{g}_{ij}^* x_j^* \neq a_i$. Then, a contradiction arises since player i can profitably increase her payoff by increasing (decreasing) x_i^* if $f_i'(x_i^* + \sum_{j \in N} \bar{g}_{ij}^* x_j^*)$ is strictly larger (smaller) than $c = f_i'(a_i)$, i.e., $x_i^* + \sum_{j \in N} \bar{g}_{ij}^* x_j^*$ is strictly smaller (larger) than a_i. This concludes the proof of Lemma 1. ■

Proof of Theorem 1. First note that as $f_j(\cdot)$ is twice continuously differentiable, strictly concave and increasing, both $f_j'(\cdot)$ and its inverse exist for all $j \in N$. If $k \leq \bar{k}$, the following is an equilibrium: $x_1^* = a_1$, while $g_{j1}^* = \min\{1, (f_j')^{-1}(k)/a_1\}$ and $x_j^* = 0$ for all players $j \in N \setminus \{1\}$;

note that, since $k \leq ca_1$, and thus, $f_j'(a_1 g_{j1}^*) = k \leq c = f_j'(a_j)$, we get, by inverting $f_j'(\cdot)$, $a_1 g_{j1}^* = (f_j')^{-1}(k) \geq (f_j')^{-1}(c) = a_j$. This equilibrium is also sociable, proving existence.

If a player j is active, there is no player i with $x_i^* > k/c$ such that $\bar{g}_{ji}^* < 1$. If not, j would have a profitable deviation by setting $g_{ji}' = g_{ji}^* + \epsilon$ accessing ϵx_i^* additional public good at a cost ϵk instead of $\epsilon c x_i^*$. Furthermore, if a player j is active, there is no player i with $x_i^* = k/c$ such that $\bar{g}_{ji}^* < 1$. Indeed, by the reasoning above, $\bar{g}_{ji}' = \bar{g}_{ji}^* + x_j^*/x_i^*$ would give i the same payoffs as the current strategy. Hence, in a sociable equilibrium, either $x_j^* = 0$ or $\bar{g}_{ji}^* = 1$.

If there are j and i such that $g_{ji}^* = 1$, then there is no z with $x_z^* \geq x_i^*$ and $\bar{g}_{zi}^* = 0$. If not, since $g_{ji}^* = 1$, $k \leq c x_i^*$, z could reduce effort by x_i^* linking to i and either strictly profit from the deviation, or weakly so, contradicting that g^* is sociable Nash equilibrium. Therefore, any i receiving active in-links is connected to all $j \in N$ such that $x_j^* \geq x_i^*$. Hence, all $i \in N$ with $x_i^* \geq k/c$ form the core $\mathcal{C}(\bar{g}^*)$.

Note that a player among those of the best type needs to be in the core; wlog, denote this player by 1. Suppose *ad absurdum* that there is $i \in \mathcal{C}(\bar{g}^*)$ with $a_i < a_1$ and $1 \notin \mathcal{C}(\bar{g}^*)$. As 1 needs to collect more public good than any player in the core, we know that $g_{1j}^* = 1$ for all $j \in \mathcal{C}(\bar{g}^*)$ (including i) and $x_1^* > 0$. However, this implies that $x_i^* + \sum_{j \in N} \bar{g}_{ij}^* x_j^* \geq x_1^* + \sum_{j \in N} \bar{g}_{1j}^* x_j^* = a_1$. By Lemma 1, i would then have an incentive to reduce her provision of the public good, a contradiction. This proves $1 \in \mathcal{C}(\bar{g}^*)$. Hence, $\mathcal{C}(\bar{g}^*) \neq \emptyset$.

As for $p \in \mathcal{P}(\bar{g}^*)$, two cases apply. If $x_p^* > 0$, we have just shown that $g_{pi}^* = 1$ for all $i \in \mathcal{C}(\bar{g}^*)$, proving *(ii)*. If $x_p^* = 0$ and $g_{pi}^* > 0$ for some $i \in \mathcal{C}(\bar{g}^*)$, the statement follows. Hence, $\bar{g}^*$ is a core-periphery graph.

Finally, if $z \in N$ with $g_{zi}^* = 0$ for all $i \in N \setminus \{z\}$, $x_z^* = a_z$. Suppose z deviates to $g_{zi}' = \varepsilon > 0$ for $i \in \arg\max x_j^*$ and $x_z' = x_z^* - \varepsilon x_i^*$. Then, the cost of attaining εx_i^* goes from $c\varepsilon x_i^*$ to $k\varepsilon$. If $x_i^* > k/c$, the deviation is strictly profitable; if $x_i^* = k/c$, the deviation is weakly profitable; hence, $g_{zi}^* = 0$ would not be part of a sociable equilibrium. This concludes the proof of Theorem 1. ■

Proof of Proposition 1. Ad *(i)*. By Lemma 1, any active player $i \in N$ consumes a_i. By Theorem 1.*(ii)*, active players form a complete core-periphery graph where all links are of weight 1. Hence, all players in the core attain the same consumption of the public good, and are of the same type. Furthermore, Theorem 1.*(ii)* implies that core players consume more public good than all other players. Hence, they have to be of type 1.

Ad *(ii)*. For all $i \in \mathcal{C}(\bar{g}^*)$, $x_i^* \geq k/c$. If $|\mathcal{C}(\bar{g}^*)| > |a_1|$, then for some player $j \in \mathcal{C}(\bar{g}^*)$, $a_j < a_1$, contradicting *(i)*. If $|\mathcal{C}(\bar{g}^*)| > \lfloor ca_1/k \rfloor$, then $x_i^* < k/c$ at least for some $i \in \mathcal{C}(\bar{g}^*)$, a contradiction. Therefore, $|\mathcal{C}(\bar{g}^*)| \leq \min\{\lfloor ca_1/k \rfloor, |a_1|\}$.

Ad *(iii)*. Suppose not and *ad absurdum* that $\sum_{i \in N} x_i^* \neq a_1$. By Lemma 1 and Theorem 1.*(ii)*, some best type either can improve her payoff by providing more public good if $\sum_{i \in N} x_i^* < a_1$, or by providing less if $\sum_{i \in N} x_i^* > a_1$, a contradiction. This concludes the proof of Proposition 1. ■

Proof of Proposition 2. Suppose that $a_i > a_j$ for any $i, j \in N$. We first show that in this case $x_i^* \geq x_j^*$. If $i \in \mathcal{C}(\bar{g}^*)$, then $a_i \equiv a_1$ and this trivially implies that $j \in \mathcal{P}(\bar{g}^*)$ and $x_i^* > x_j^*$. If $i, j \in \mathcal{P}(\bar{g}^*)$, suppose *ad absurdum* that $x_i^* < x_j^*$. Then, given that $x_j^* > 0$, $\bar{g}_{jz}^* = 1$ for all $z \in \mathcal{C}(\bar{g}^*)$ and since $a_i > a_j$ also $\bar{g}_{iz}^* = 1$. Hence, $\sum_{z\in N} \bar{g}_{iz}^* x_z^* = \sum_{z\in N} x_z^* = \sum_{z\in N} \bar{g}_{jz}^* x_z^*$. However, since $x_i^* < x_j^*$, i receives less public good than j, and a contradiction arises with $a_i > a_j$. Therefore, $x_i^* \geq x_j^*$.

Now we show that $\sum_{z\in N} \bar{g}_{iz}^* \geq \sum_{z\in N} \bar{g}_{jz}^*$. If $i \in \mathcal{C}(\bar{g}^*)$ and $j \in \mathcal{P}(\bar{g}^*)$, then trivially $\sum_{z\in N} \bar{g}_{iz}^* \geq \sum_{z\in N} \bar{g}_{jz}^*$. If $i, j \in \mathcal{P}(\bar{g}^*)$, suppose *ad absurdum* that $\sum_{z\in N} \bar{g}_{iz}^* < \sum_{z\in N} \bar{g}_{jz}^*$. This implies that $\sum_{z\in N} g_{iz}^* < \sum_{z\in N} g_{jz}^*$. Since i, j link to the largest contributors in the core, this implies that $\sum_{z\in N} g_{iz}^* x_z^* < \sum_{z\in N} g_{jz}^* x_z^*$. Moreover, as we have just shown, $x_i^* \geq x_j^*$. This yields a contradiction with $a_i > a_j$, or it implies that either i (by linking more and providing less public good) or j (by linking less) can deviate profitably. Hence, $\sum_{z\in N} g_{iz}^* \geq \sum_{z\in N} g_{jz}^*$.

Suppose now that $a_i > a_j$ for any $i, j \in N$ and $x_i^* \geq x_j^* > 0$. Then, $g_{iz}^* = g_{jz}^* = 1$ for all $z \in \mathcal{C}(\bar{g}^*)$, and thus i and j receive the same spillovers from the network. Then, $x_i^* > x_j^*$ follows from $a_i > a_j$. This concludes the proof of Proposition 2. ■

Proof of Proposition 3. Ad *(i)*. Suppose that $x_i^* > x_j^*$ for some $i, j \in \mathcal{C}(\bar{g}^*)$. By Proposition 1.*(i)*, $a_i = a_j = a_1$ or equivalently, $\sum_{z\in N} \bar{g}_{iz}^* x_z^* + x_i^* = \sum_{z\in N} \bar{g}_{jz}^* x_z^* + x_j^*$, which trivially implies $\sum_{z\in N} \bar{g}_{iz}^* x_z^* < \sum_{z\in N} \bar{g}_{jz}^* x_z^*$. Moreover, given that $x_i^* > x_j^*$, $\sum_{z\in N} \bar{g}_{iz}^* \geq \sum_{z\in N} \bar{g}_{jz}^*$.

Ad *(ii)*. Suppose that $x_i^* > 0$ for i such that $a_i < a_1$ and *ad absurdum* that some $j \notin \mathcal{C}(\bar{g}^*)$ with $a_j = a_1$. Then, $i, j \in \mathcal{P}(\bar{g}^*)$ and thus $\bar{g}_{ij}^* = 0$, while $x_i^* > 0$ implies that $g_{iz}^* = 1$ for all $z \in \mathcal{C}(\bar{g}^*)$, and a contradiction arises with $a_j = a_1$ since any $z \in \mathcal{C}(\bar{g}^*)$ receives x_i^*, while j does not, and by Proposition 1.*(i)*, $a_z = a_1$.

Ad *(iii)*. By Theorem 1.*(ii)* active players form a complete core-periphery graph, and thus, a player j is active if, and only if, $\sum_{i\in\mathcal{C}(\bar{g}^*)} x_i^* < a_j$. All players for whom $a_j \leq \min_{s^*\in SNE} \sum_{i\in\mathcal{C}(\bar{g}^*)} x_i^*$ are inactive, where SNE is the set of all sociable Nash equilibria. Then, picking the smallest such $i \equiv \tilde{n}$, $x_i^* = 0$ for all such i.

Ad *(iv)*. Given $x_i^* = 0$ and $a_i^* < a_1$ implies that $g_{iz}^* > 0$ for some $z \in \mathcal{C}(\bar{g}^*)$. This in turn implies that $cx_z^* \geq k$. Deriving i's FOC shows that it is beneficial for i to sponsor weighted links until $\sum_{z\in N} g_{iz}^* x_z^* = (f_i')^{-1}(k) \geq (f_i')^{-1}(c) = a_i$ given that $x_z^* \geq k/c$ for all $z \in \mathcal{C}(\bar{g}^*)$. This concludes the proof of Proposition 3. ■

Proof of Proposition 4. Take a strict Nash equilibrium and a player p such that $x_p^* = 0$, which implies that $p \in \mathcal{P}(\bar{g}^*)$. Suppose *ad absurdum* that $g_{pi}^* \in (0,1)$ and $g_{pj}^* \in (0,1)$ for $i, j \in \mathcal{C}(\bar{g}^*)$, where, wlog, $x_i^* > x_j^*$. Then, there is $\epsilon > 0$ such that $g_{pi}' = g_{pi}^* + \epsilon$ and $g_{pj}' = g_{pj}^* - \epsilon$ is a profitable deviation. Hence, for any $j, p \in N$, if $g_{pj}^* > 0$, then $g_{pi}^* = 1$ for all $i \in N$ such that $x_i^* > x_j^*$. This implies that $\bar{g}^*$ is a weighted nested split graph, concluding the proof of Proposition 4. ■

Proof of Proposition 5. First note that the result of Lemma 1 extends to the model with $l(\cdot)$ as it does not depend on the linking technology. Note also that, following the same reasoning as in Theorem 1, if $a_i > a_j$ for $i, j \in N$, $\eta_i(\bar{g}) > \eta_j(\bar{g})$; otherwise, j would consume more public good than i, a contradiction. Additionally, if $g^*_{ij} > 0$ and there is a player z with $x^*_z > x^*_j$ for $i, j, z \in N$, then $\bar{g}^*_{iz} > \bar{g}^*_{ij}$; otherwise i would have a profitable deviation by reducing g^*_{ij} while increasing g^*_{iz}. Also remember that in a sociable equilibrium all weakly profitable links are established.

A player i's maximization problem is given by:

$$\max_{e_i, x_i} f_i\left(x_i + \sum_{z \in N} \bar{g}_{iz} x^*_z\right) - c(x_i) - k \sum_{z \in N} e_{iz},$$

with $x_i = \max\{a_i - \sum_{z \in N} \bar{g}_{iz} x^*_z, 0\}$.

If i is active, by Lemma 1, i's consumption of the public good in any equilibrium is a_i. Hence, small changes in i's spillovers translate into small changes of her provision, but not her consumption in equilibrium. As a result, i's maximization problem can be written as

$$\max_{e_i} f_i(a_i) - c\left(a_i - \sum_{z \in N} \bar{g}_{iz} x^*_z\right) - k \sum_{z \in N} e_{iz}.$$

As by definition $\bar{g}_{ij} = \max\{l(e_{ij}), l(e_{ji})\}$ for any $j \in N \setminus \{i\}$, the FOC with respect to e_{ij} gives

$$\frac{\partial \max\{l(e_{ij}), l(e_{ji})\}}{\partial e_{ij}} c x^*_j \leq k, \tag{A-1}$$

with equality if $e^*_{ij} > 0$. Indeed, as $k > 0$, if $l(e_{ij}) \leq l(e_{ji})$, $\bar{g}_{ij}$ is determined by $l(e_{ji})$, and the LHS of (A-1) is null; hence, in equilibrium $e^*_{ij} = 0$. If instead $l(e_{ij}) > l(e_{ji})$, $\bar{g}_{ij}$ is determined by $l(e_{ij})$; since $l(e) < 1$ for any finite e, $l(1)$ is not a possible corner solution; as a result, i will invest in this link up to the point where the marginal cost equals the marginal benefit. This implies that $g^*_{ij} = g^*_{zj}$ for all $i, j, z \in N$ with $x^*_j > x^*_i, x^*_z > 0$.

If i is inactive, i's maximization problem can be written as

$$\max_{e_i} f_i\left(\sum_{z \in N} \bar{g}_{iz} x^*_z\right) - k \sum_{z \in N} e_{iz}.$$

Since i is inactive, in equilibrium no player links to i, i.e., $l(e_{ji}) = 0$ for all $j \in N$. Hence, the FOC with respect to e_{ij} gives

$$f'_i\left(\sum_{z \in N} g_{iz} x^*_z\right) \frac{\partial l(e_{ij})}{\partial e_{ij}} x^*_j \leq k, \tag{A-2}$$

with equality if $e_{ij} > 0$. Indeed, if j's provision is not large enough with respect to the linking cost k, the LHS is lower than the RHS, and i would find it optimal to set $e_{ij} = 0$. If instead $e_{ij} > 0$, e_{ij} is set for marginal cost to be equal to the marginal benefits of that link, given that $l(e_{ij}) < 1$ for any finite e_{ij}.

To prove that $\bar{g}^*$ is a weighted core-periphery graph, it is left to prove that, if there are players i and j such that $g^*_{ij} > 0$, then there is no player $z \in N$ with $x^*_z \geq x^*_j > 0$ and $\bar{g}^*_{jz} < \bar{g}^*_{ij}$. To see this, comparing (A-1) and (A-2) reveals that $\bar{g}^*_{jz} \geq \bar{g}^*_{ij}$ if $\bar{g}^*_{jz} > 0$. Furthermore, if $\bar{g}^*_{jz} = 0$ and $x^*_z > x^*_j > 0$, j could set $e'_{jz} = \epsilon$, thereby reducing public good provision by $g(\epsilon)x^*_z$; by (A-1), there is an $\epsilon > 0$ for which this deviation is strictly profitable, in which case, in a sociable equilibrium, $e'_{jz} > 0$. If instead $x^*_z = x^*_j > 0$ and $\bar{g}^*_{jz} = 0$, the deviation is weakly profitable; hence, in a sociable equilibrium, either $x^*_j = 0$ or $\bar{g}^*_{jz} > 0$. These arguments also imply that $g^*_{iz} = g^*_{jz}$ for all $i, j, z \in N$ with $x^*_i, x^*_j \in (0, x^*_z)$. This concludes the proof of Proposition 5. ■

Proof of Corollary 1. Take a sociable Nash equilibrium (x^*, g^*), and wlog, assume $i < j$ if $x^*_i \geq x^*_j$. First, note that if a player i has a weighted link to j, then i is inactive. To show this, suppose that i has a weighted link to j and i is active. Lemma 1 implies $x_i + \sum_{l \in N} \bar{g}_{il} x^*_l = a_i$. Then, i linking with a weight $g^*_{ij} < 1$ implies that $x^*_j \geq K/(cg^*_{ij}) + k/c > k/c$. On the contrary, setting g'_{ij} and x'_i such that $x'_i + \sum_{i \in N_i(g')} g'_{ij} x^*_j = a_i$ is not profitable if $x^*_j < k/c$, a contradiction. Hence, either i has no weighted links, or i is inactive. Following similar arguments as in Theorem 1, g^* is a weighted core-periphery graph where active players have links of weight 1.

Second, if g^* is empty, the statement trivially follows. Otherwise, the most profitable link is to player 1. Now consider a scalar $\alpha \in \mathcal{R}^+$ and introduce a player z such that $U_z(x, g) = \alpha f_1\Big(x_z + \sum_{j \in N} \bar{g}_{zj} x_j\Big) - cx_z - \sum_{j \in N_z(g)} (K + kg_{zj})$; i.e., z has the same benefits as 1, but scaled down by α. In particular, assuming that α is low enough that when $g_{z1} = 1$, z is inactive, z is isolated if $\alpha f_1(g_{z1} x^*_1) - kg_{z1} - (\alpha f_1(a_z) - ca_z) < K$, where $g_{z1} \in (0, 1]$ is the optimal weight of the link to 1 chosen by z. Using the envelope theorem, the derivative of the LHS with respect to α is $f_1(g_{z1} x^*_1) - f_1(a_z) > 0$, so that the LHS is continuously differentiable and increasing in α, and equal to 0 if $\alpha = 0$. Hence there is $\underline{\alpha} > 0$ such that z is isolated for all $\alpha \in (0, \underline{\alpha})$. Following a similar argument, if $x^*_2 \geq k/c$ and $x^*_2 < x^*_1$, there is $\overline{\alpha} > 0$ such that $g^*_{z1} = 1$ but $g^*_{z2} = 0$ for all $\alpha \in [\underline{\alpha}, \overline{\alpha})$. This concludes the proof of Corollary 1. ■

Proof of Corollary 2. Since (x^*, g^*) is a strict Nash equilibrium when $K = 0$, for any $i \in \mathcal{C}(\bar{g}^*)$ and $j \in N$ such that $x^*_i > 0$ and $g^*_{ji} = 1$, $x^*_i > k/c$. Suppose now $K > 0$ and denote by (x', g') an equilibrium. Then, note first that if $g'_{ij} = g^*_{ji} = 1$, $x'_i = x^*_i$ for any i, j such that $x^*_i, x^*_j > 0$, as K does not affect the marginal cost of consuming the public good directly or indirectly. As $x^*_i > k/c$, there exists κ_i such that $x^*_i = (\kappa_i + k)/c$ so that $x^*_i > (K + k)/c$ for any $K < \kappa_i$ and $g'_{ji} = 1$. For a player j such that $x^*_j = 0$, consider j's least profitable link in g^*, call it g^*_{jz}. Then,

g'_{jz} solves $x'_z f'_j(\sum_{i\in N} x'_i) = k$. As this condition does not depend on K and $x'_i = x^*_i$ for any player in $i \in \mathcal{C}(\bar{g}^*)$, $g^*_{jz} = g'_{jz}$ if $g'_{jz} > 0$. Finally, $g'_{jz} > 0$ if

$$x^*_z\left(1 - \frac{K}{c(f')^{-1}}\left(\frac{k}{x^*_z}\right)\right) > \frac{k}{c}. \tag{A-3}$$

Clearly the condition is satisfied for $K = 0$. Furthermore, the LHS is strictly decreasing in K. Hence, there exists κ_j such that (A-3) is satisfied for any $K < \kappa_j$.
Define $\kappa = \min_{i\in\mathcal{C}(\bar{g}^*), j\in N|x^*_j=0}\{\kappa_i, \kappa_j\}$. As a result, for any $K < \kappa$, $(x^*, g^*) = (x', g')$. This concludes the proof of Corollary 2. ■

Proof of Corollary 3. First we show that a non-empty network is a weighted core-periphery graph. If a player j is active, there is no player i with $x^*_i > k/c$ such that $g^*_{ji} < 1$. If not, as in the two-way flow model, j would have a profitable deviation by setting $g'_{ji} = g^*_{ji} + \epsilon$ accessing ϵx^*_i additional public good at a cost ϵk instead of $\epsilon c x^*_i$. Since $x^*_i > k/c$, the deviation is profitable. If instead $x^*_i = k/c$, $g^*_{ji} < 1$ is not part of a sociable equilibrium. So all players producing more than k/c are linked with each other and form the core. Periphery players link to the highest producers, so that a sociable Nash equilibrium network is a weighted core-periphery graph where active players have links of weight 1.
Finally, given any equilibrium (x^*, g^*), player j is in the periphery if $a_j - \sum_{i\in\mathcal{C}(g^*)} x^*_i < k/c$. Define by NE the set of all Nash equilibria. Then, for all $j \geq \underline{n}$, $j \in \mathcal{P}(g^*)$ given any equilibrium for which $a_j - \min_{s^*\in NE}\sum_{i\in\mathcal{C}(g^*)} x^*_i < k/c$. This concludes the proof of Corollary 3. ■

References

Allouch, N., 2015. On the private provision of public goods on networks. J. Econ. Theory 157, 527–552. doi:/10.1016/j.jet.2015.01.007.

Allouch, N., 2017. The cost of segregation in (social) networks. Games Econ. Behav. 106, 329 – 342. doi:10.1016/j.geb.2017.08.012.

Bala, V., Goyal, S., 2000. A noncooperative model of network formation. Econometrica 68, 1181–1229. doi:10.1111/1468-0262.00155.

Baumann, L., 2021. A model of weighted network formation. Theor. Econ. 16, 1–23. doi:10.3982/TE2177.

Bergstrom, T., Blume, L., Varian, H., 1986. On the private provision of public goods. J. Public Econ. 29, 25–49. doi:10.1016/0047-2727(86)90024-1.

Billand, P., Bravard, C., Sarangi, S., 2011. Strict Nash networks and partner heterogeneity. Int. J. Game Theory 40, 515–525. doi:10.1007/s00182-010-0252-8.

Bloch, F., Dutta, B., 2009. Communication networks with endogenous link strength. Games Econ. Behav. 66, 39–56. doi:10.1016/j.geb.2008.03.007.

Bloch, F., Jackson, M.O., 2007. The formation of networks with transfers among players. J. Econ. Theory 133, 83–110. doi:10.1016/j.jet.2005.10.003.

Bramoullé, Y., Kranton, R., 2007. Public goods in networks. J. Econ. Theory 135, 478–494. doi:10.1016/j.jet.2006.06.006.

Cabrales, A., Calvó-Armengol, A., Zenou, Y., 2011. Social interactions and spillovers. Games Econ. Behav. 72, 339–360. doi:10.1016/j.geb.2010.10.010.

Conley, T., Udry, C., 2010. Learning about a new technology: Pineapple in Ghana. Am. Econ. Rev. 100, 35–69. doi:10.1257/aer.100.1.35.

Ding, S., 2019. A factor influencing network formation: Link investment substitutability. doi:10.2139/ssrn.3502222. Mimeo.

Feick, L., Price, L., 1987. The market maven: A diffuser of marketplace information. J. Mark. 51, 83–97. doi:10.1177/002224298705100107.

Galeotti, A., Goyal, S., 2010. The law of the few. Am. Econ. Rev. 100, 1468–1492. doi:10.1257/aer.100.4.1468.

Galeotti, A., Goyal, S., Kamphorst, J., 2006. Network formation with heterogeneous players. Games Econ. Behav. 54, 353–372. doi:10.1016/j.geb.2005.02.003.

Galeotti, A., Merlino, L.P., 2014. Endogenous job contact networks. Int. Econ. Rev. (Philadelphia) 55, 1201–1226. doi:10.1111/iere.12087.

Haller, H., Kamphorst, J., Sarangi, S., 2007. (Non-) existence and scope of Nash networks. Econ. Theory 31, 597–604. doi:10.1007/s00199-006-0105-7.

Hiller, T., 2017. Peer effects in endogenous networks. Games Econ. Behav, 105, 349–367. doi:10.1016/j.geb.2017.07.010.

Jackson, M.O., 2010. Social and Economic Networks. Princeton University Press, Princeton. doi:10.1515/9781400833993.

Jackson, M.O., Wolinsky, A., 1996. A strategic model of social and economic networks. J. Econ. Theory 71, 44–74. doi:10.1006/jeth.1996.0108.

Kinateder, M., Merlino, L.P., 2017. Public goods in endogenous networks. Am. Econ. J. Microecon. 9, 187–212. doi:10.1257/mic.20140276.

Kinateder, M., Merlino, L.P., 2021. Free riding in networks. URL: `https://arxiv.org/abs/2110.11651`. Mimeo.

Mahadev, N.V., Peled, U.N., 1995. Threshold graphs and related topics. Elsevier, Amsterdam.

Merlino, L.P., 2014. Formal and informal job search. Econ. Lett. 125, 350–352. doi:10.1016/j.econlet.2014.10.008.

Merlino, L.P., 2019. Informal job search through social networks and vacancy creation. Econ. Lett. 178, 82–85. doi:10.1016/j.econlet.2019.03.006.

Warr, P.G., 1983. The private provision of a public good is independent of the distribution of income. Econ. Lett. 13, 207–211. doi:10.1016/0165-1765(83)90087-3.