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Nonparametric Estimation of Convergence of Interest Rates: Effects on Bond Pricing

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ABSTRACT

We present and estimate a model of short term interest rate dynamics where we incorporate the convergent behavior of interest rates implied by the transition to EMU. We apply this model to data of two EMU countries -Spain and Italy- and compare the performance, in terms of accuracy of bond pricing, of this two-factor convergence model with alternative specifications. Nonparametric techniques are used for the estimation of the processes. The two-factor model which accounts for the convergence with Europe of the domestic economies, obtains better results than alternative models mainly for short-term assets. The results of the nonparametric specifications are shown to be significantly better than those of parametric alternatives.

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1 Introduction

The behavior of the term structure of interest rates has attracted increased attention during the last decade, because of the implications for the correct pricing of fixed income securities and derivatives. While the effort has been placed in developing models that would yield closed form solutions for the prices of interest rate derivatives, little attention has been paid to the correctness of the interest rate dynamics implied by the specified model. This is especially worrisome since some models do not replicate the most common stylized facts of interest rate behavior: level effects in volatility –the variance usually being more than linear on the level–, near integrated behavior or high persistence in term spreads.[1] As a consequence, misspecification of the underlying interest rate model leads to serious pricing errors (Canabarro, 1995, Backus et al. 1998) and to distorting effects on the allocation of investment.

Recent contributions that solve some of these deficiencies include Aït-Sahalia (1996 a), Brenner et al. (1996), Tauchen (1996), Koedijk et al (1997), Bali (2000, 2003) and Boudoukh et al. (2000). A main finding common to these papers, qualified by Aït-Sahalia (1996b) and Jones (2003), is the very weak mean reversion of interest rates –near integration– that suggests the existence of nonlinearities in the mean reverting behavior. Additionally, it appears critical to model volatility correctly since its effects on the pricing of assets are of substantial magnitude. In the case of interest rates, volatility depends on the level of the interest rate (Chan et al., 1992, hereafter CKLS), and this dependence appears to be stronger than some traditional models such as that of Cox et al. (CIR, 1985) have specified.

[1] Pagan et al. (1996) provide an excellent summary of these stylized facts and of how most of the existing models fail to account for them.

Additionally, a number of papers have proposed new estimation methods that can handle the complexity of the dynamic properties of interest rate data. Robust estimation techniques (semiparametric or nonparametric) have been suggested to avoid the arbitrary restrictions imposed by pre-specified parametric functional forms. Examples are Aït-Sahalia (1996 a,b), Boudoukh et al. (1997, 2000), Stanton (1997), Ghysels and Ng (1998), Downing (1999) and Broadie et al. (2000).

In this article we specify a model of interest rate dynamics that attempts to capture a distinctive feature of European interest rates, namely the convergent behavior during the years prior to the creation of the European Monetary Union (EMU). Building on the special characteristics of the transition process to EMU we develop a two-factor model of the domestic interest rate of a European country where we include a European short-term interest rate as a second factor:[2] the domestic rate follows a mean reverting process that reverts to a stochastic mean which is identified with the European rate. We use nonparametric regression techniques to estimate the short-term interest rate processes in two European countries: Spain and Italy. In both cases we find evidence of a substantial level effect in volatility and –somewhat weaker– nonlinear drifts.[3]

Since the dynamic behavior of interest rates affects the prices of bonds and of other interest rates derivatives, incorporating a second factor should improve the pricing of these assets. Once we have estimated the interest rate processes and calculated

[2] Most models characterize interest rates as one-factor Markovian processes. Theoretical and empirical research have suggested that two-factor models describe the behavior of the term structure better than one-factor models. The second factor is usually identified with the volatility (Longstaff and Schwartz, 1992) or the mean rate (Balduzzi et al., BDF 1998, or Balduzzi et al., BDFS 2000). Some studies (Chacko 1997, Downing 1999 and BDFS 2000) also consider three or four factors.

[3] Similar empirical results can be found in CKLS (1992), Gallant and Tauchen (1996), Pfann et al. (1996) and Jones (2003).

the market price of risk we proceed to value zero coupon bonds using simulation. We compare the bond prices obtained by our methodology with actual prices. As benchmark for comparison, we also estimate a one-factor model –similar to that in Stanton (1997)– where the dynamics of the short term rate depend only on its own level. Our specification obtains pricing errors that are smaller both because of the model structure –around 5% for one year bonds and 2% for two year bonds– and because of the estimation technique –around 30%. We offer some comments as to how the performance of our model might compare with alternative two-factor models.

The rest of the paper is organized as follows: in Section 2 we present the convergence model and the methods used for the estimation of the interest rate processes. We show results for Spain, Italy and for a European rate. In Section 3 we derive the necessary formulas to estimate the market price of European and domestic interest rate risk. In Section 4 we present the simulations used to price zero coupon bonds and the results of the pricing exercise. Finally in Section 5 we conclude.

2 Interest rates in Europe: The Convergence Model

Mean reverting models for the term structure have enjoyed wide acceptance since Vasicek (1977) and CIR (1985). Both these models are one-factor models with a constant long run mean that cannot account for the behavior of the term structure: in a one-factor world the prices of assets that depend on that factor should be perfectly correlated. Furthermore, the evolution of most series of interest rates in the last years shows that the assumption of a constant long run mean seems unjustifiable. Consequently, a natural extension would be to include as an additional factor a stochastic mean. In particular, in this paper we model the short term interest rate

of small European countries as a mean reverting process where the mean changes stochastically over time. We consider this framework adequate to study the term structure of interest rates of EMU countries. The Maastricht Treaty required interest rates of domestic countries to converge to a level given by the rates of the countries with lowest inflation levels. We argue that a two-factor model of the interest rate, where the factors are the level of the domestic short term rate and the central tendency of that short term rate is an adequate representation of the behavior of interest rates in EMU candidates –and, after EMU creation, of EMU members.[4] We identify the stochastic mean of the interest rate with an observable European rate. This is an important difference with other stochastic mean reverting models, that leave the mean unidentified and have to estimate it from the data.

Assume that the interest rates of two countries d and e evolve independently except for, maybe, a correlation in the innovations to their processes. Each rate reverts to its own mean, θ_d and θ_e. Specifying the dynamics as two linear mean reverting processes, the behavior of the two rates would correspond to

$$\begin{aligned} dr_d &= \kappa_d \left(\theta_d - r_d\right) dt + \sigma_d dz_d \\ dr_e &= \kappa_e \left(\theta_e - r_e\right) dt + \sigma_e dz_e \end{aligned} \tag{1}$$

where the κ_i terms represent the speed of adjustment to the mean, the volatility terms σ_i could be a constant ($\sigma_i = \sigma$) or a volatility in levels "a la CKLS (1992)" ($\sigma_i = \sigma r_i^{\gamma}$) and the dz's are possibly correlated. Now assume that the rate of country

[4] The setup of EMU does not force domestic interest rates to be the same across member countries: Domestic market rates in a currency union should approximately be the same but they may differ because of sovereign or credit risk. This difference may become more notorious for medium and long term debt. Also, differences in default risk and tax treatment –fiscal law has not been unified in EMU– imply that some individual bond markets may remain segmented so it is likely that differences in rates and in rate dynamics still remain.

d follows a convergent behavior towards the rate in country e: r_d is trying to get closer to –to "converge to"– r_e so at each point in time the relevant equilibrium value for r_d is the current value of r_e, which therefore becomes a stochastic mean for r_d. The two diffusions in (1) become:

$$\begin{aligned} dr_d &= \kappa_d (r_e - r_d)\, dt + \sigma_d dz_d \\ dr_e &= \kappa_e (\theta_e - r_e)\, dt + \sigma_e dz_e \end{aligned} \tag{2}$$

This reasoning directly applies to countries that were candidates to EMU, in the years prior to the adoption of the single currency. Given the requirements of the Maastricht Treaty the rates in domestic countries evolved with reference to a "European" rate given by the average of the rates of some countries.

We could now estimate the model as it is. Following some continuous-time technique –Hansen and Scheinkman's (1995) method, simulation based methods such as indirect inference (Gourieroux et al., 1993) or the Efficient Method of Moments (Gallant and Tauchen, 1996), exact maximum likelihood of the discrete data (Fernández-Navas, 1999, Gourieroux and Jasiak, 2002)– estimates of the parameters of the two processes could be obtained. However, linear processes such as (2) perform quite badly empirically (Pagan et al., 1996). Significant nonlinearities in mean reversion of interest rates are apparent so a specification that imposes linearity becomes too restrictive. One would like to specify a general form for the drift and volatility terms that was flexible enough to account for possible nonlinearities. We follow this approach, and leave both the drift and the volatility functions unspecified. We therefore postulate that the short term domestic interest rate in EMU countries, r_d, follows a

process given by the diffusion

$$dr_d = \mu_d(r_e - r_d)dt + \sigma_d(r_d)dz_d \tag{3}$$

and the mean reverting level, r_e, is the European interest rate which evolves over time following

$$dr_e = \mu_e(r_e)dt + \sigma_e(r_e)dz_e \tag{4}$$

dz_d and dz_e may be correlated with correlation coefficient ρ.[5] $\mu(\cdot)$ and $\sigma(\cdot)$ are general nonparametric forms for the drift and diffusion functions of the two processes.[6]

We call equations (3) and (4) *Model 1.* Given that we did not specify a functional form for both the drift and volatility functions, we need to provide a *parameter-free* estimation of our model using nonparametric techniques. Since the data are only available in discrete time we briefly outline in Section 2.1 the procedure we follow to estimate the underlying continuous-time functions with discrete data. Section 2.2. describes the estimation technique.

[5]The volatility of the domestic rate could be assumed to react also to the level of the European rate or to the rate differential:

$$\begin{aligned} (1)\ dr_d &= \mu_d(r_e - r_d)dt + \sigma_d(r_d, r_e)dz_d \\ (2)\ dr_d &= \mu_d(r_e - r_d)dt + \sigma_d(r_d, r_e - r_d)dz_d \end{aligned}$$

We have explored the second specification, more reasonable in our context, and the relevant results did not change much. However, when using two conditioning variables for the nonparametric estimation of the volatility we ran into the curse of dimensionality, given that it is in the high values of the spread $r_e - r_d$ where its effect in volatility appears but for those values the density of both the spread and the domestic rate is lower. Thus, an empty-space phenomenon arises that impedes a good estimation of the bivariate specification of the volatility. We opted for keeping only r_d, the conditioning variable that had a clearer effect on volatility.

[6]We assume that $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the smoothness conditions required for the analysis of the stochastic process (Duffie, 2001) and for the posterior nonparametric estimation (Pagan and Ullah, 1999).

2.1 Approximations to the continuous time functions

The procedure we use to obtain estimations of the nonparametric drift and volatility functions –based on Stanton (1997)– relies on approximating the infinitesimal generator of the continuous-time functions in the interest rate dynamics. This methodology stems from Hansen and Scheinkman (1995) where the infinitesimal generators of the drift and volatility of a continuous-time parametric diffusion process are used to generate moment conditions from which the parameters can be estimated via GMM. Given our nonparametric specification for the drifts and volatilities, the methodology of Hansen and Scheinkman cannot be directly applied in our context. Stanton (1997) modified the procedure so that it could be used to estimate nonparametric drifts and diffusions by relying on stochastic Taylor-series expansions of the infinitesimal generator of functions related to the diffusion process studied. Given that he used a one-factor model, we need to extend and complement his analysis so that it can be applied to our two-factor specification of the evolution of the domestic interest rate.

Assume a general multivariate diffusion process X with k elements

$$\begin{aligned} dX &= \mu(X)dt + \sigma(X)dZ \\ corr(dZ) &= R \end{aligned} \tag{5}$$

where $\mu(X)$ and $\sigma(X)$ are $k \times 1$ vectors that contain the drift and diffusion terms, dZ is a k dimensional Wiener process and R is a $k \times k$ matrix with ij coefficient ρ_{ij}. Take now any smooth function of that process, $f(x,t)$. The infinitesimal generator of that function, which gives the expected change in $f(x,t)$ in the next infinitesimal

period of time, is defined as

$$\begin{aligned}\Im f(x,t) &= \lim_{\tau\downarrow t}\frac{E_t[f(X_\tau,\tau)|X_t=x]-f(x,t)}{\tau-t} = \\ &= \frac{\partial f(x,t)}{\partial t}+\sum_{i=1}^{k}\frac{\partial f(x,t)}{\partial x_i}\mu_i(x)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{\partial^2 f(x,t)}{\partial x_i\partial x_j}\sigma_i(x)\sigma_j(x)\rho_{ij}\end{aligned} \tag{6}$$

We can obtain an approximation to that infinitesimal generator by an stochastic Taylor-series expansion of the term $E_t\left[f(X_{t+\triangle},t+\triangle)\right]$ around $f(X_t,t)$ which yields:

$$E_t\left[f(X_{t+\triangle},t+\triangle)\right]=f(X_t,t)+\triangle\Im f(X_t,t)+\frac{1}{2}\triangle^2\Im^2 f(X_t,t)+...+\frac{1}{n!}\triangle^n\Im^n f(X_t,t)+O(\triangle^{n+1}) \tag{7}$$

and after rearranging terms:

$$\Im f(X_t,t)=\frac{1}{\triangle}E_t\left[f(X_{t+\triangle},t+\triangle)-f(X_t,t)\right]-\frac{1}{2}\triangle\Im^2 f(X_t,t)-...-\frac{1}{n!}\triangle^{n-1}\Im^n f(X_t,t)+O(\triangle^n) \tag{8}$$

which allows us to approximate $\Im f(X_t,t)$ arbitrarily close with an error of order $O(\triangle^n)$. An approximation of order one would be

$$\Im f(X_t,t)=\frac{1}{\triangle}E_t\left[f(X_{t+\triangle},t+\triangle)-f(X_t,t)\right]+O(\triangle) \tag{9}$$

and multiplying this by two and subtracting an expansion with $\triangle t=2\triangle$ we get a second order approximation

$$\Im f(X_t,t)=\frac{1}{2\triangle}\left\{4E_t\left[f(X_{t+\triangle},t+\triangle)-f(X_t,t)\right]-E_t\left[f(X_{t+2\triangle},t+2\triangle)-f(X_t,t)\right]\right\}+O(\triangle^2) \tag{10}$$

and so on.

Now we need to find a function $f(x,t)$ such that $\Im f(x,t)=g(x,t)$, where $g(x,t)$ is the function we want to approximate. In our case these functions are $\mu_d(r_e-r_d)$, $\sigma_d(r_d)$, $\mu_e(r_e)$, $\sigma_e(r_e)$, the scalar ρ and the two prices of risk. We outline the procedure

for the first five terms here and postpone the discussion on the market price of risk until Section 3.

- *Drifts*: Take $f(r_d, r_e, t) = r_d$. Then from (6) it is immediate to find that, given that r_d and r_e are known at time t, the infinitesimal generator of $f(r_d, r_e, t)$ is:

$$\Im f(r_d, r_e, t) = \Im\left(r_d\right) = \lim_{\tau \downarrow t} \frac{E_t[r_{d,t+\tau} | r_{d,t} = r_d, r_{e,t} = r_e] - r_d}{\tau - t} = \mu_d(r_e - r_d) \tag{11}$$

so the drift can then be approximated by

$$\mu_d\left[(r_e - r_d)_t\right] = \Im\left(r_d\right) = \frac{1}{\triangle} E_t\left[r_{d,t+\Delta} - r_{d,t}\right] + O(\triangle) \tag{12}$$

or by higher order expansions.[7] This expression is also directly applicable to the drift of the European rate $\mu_e(r_e)$, which is conditional only on the own level of the rate.

- *Volatility*: Take $f(r_d, r_e, t) = \left(r_d - r_{d,t}\right)^2$. Then the infinitesimal generator of $f(r_d, r_e, t)$ is

$$\Im f(r_d, r_e, t) = \Im\left[\left(r_d - r_{d,t}\right)^2\right] = \mu_d(r_e - r_d) \cdot 2(r_d - r_{d,t}) + \frac{1}{2} \cdot 2 \cdot \sigma_d^2(r_d) \tag{13}$$

which evaluated at $r_d = r_{d,t}$, $r_e = r_{e,t}$ yields $\Im f(r_{d,t}, r_{e,t}, t) = \sigma_d^2(r_{d,t})$. Thus, $\sigma_d^2(r_{d,t})$ can be approximated by

$$\sigma_d^2(r_{d,t}) = \Im\left[\left(r_d - r_{d,t}\right)^2\right] = \frac{1}{\triangle} E_t\left[\left(r_{d,t+\triangle} - r_{d,t}\right)^2\right] + O(\triangle) \tag{14}$$

or by higher order approximations. The same applies to the variance of the European rate, $\sigma_e^2(r_e)$.

- *Correlation coefficient*: Take $f(r_d, r_e, t) = \left(r_d - r_{d,t}\right)\left(r_e - r_{e,t}\right)$. Then the infinitesimal generator of $f(r_d, r_e, t)$ is

$$\begin{aligned} \Im f(r_d, r_e, t) &= \Im\left[\left(r_d - r_{d,t}\right)\left(r_e - r_{e,t}\right)\right] = \\ &= \mu_d(r_e - r_d) \cdot (r_e - r_{e,t}) + \mu_e(r_e) \cdot (r_d - r_{d,t}) + \frac{1}{2} \cdot 2 \cdot \sigma_d(r_d)\sigma_e(r_e)\rho(r_d, r_e) \end{aligned} \tag{15}$$

[7] Note that this first order approximation yields the naïve discretization with $\Delta = 1$.

which evaluated at $r_d = r_{d,t}$, $r_e = r_{e,t}$ yields $\Im f(r_d, r_e, t) = \sigma_d(r_{d,t})\sigma_e(r_{e,t})\rho(r_d, r_e)$, a "conditional covariance" $cov(r_d, r_e)$. We divide by the volatilities and integrate over the distribution of r_d and r_e to find the unconditional value of ρ. Thus the correlation coefficient of the two Wiener processes can be approximated by taking the expectation over r_d and r_e of the conditional estimate:

$$\rho = E\left[\frac{1}{\triangle\sigma_d(r_{d,t})\sigma_e(r_{e,t})} E_t\left[(r_{d,t+\triangle} - r_{d,t})(r_{e,t+\triangle} - r_{e,t})\right]\right] \tag{16}$$

or by higher order approximations of the term inside the conditional expectation.

In this paper we use the first order approximations to the drift, volatility and correlation coefficient terms. Stanton (1997) showed simulation evidence that with high frequency data (daily or weekly data) the first order approximation is accurate enough. Consequently, we estimate the drift and the variance from[8]

$$\mu\left[(r_e - r_d)_{t-1}\right] = E[r_{d,t} - r_{d,t-1}|r_{d,t-1}, r_{e,t-1}] + u_t = E[\Delta r_{d,t}|r_{d,t-1}, r_{e,t-1}] + u_t \tag{17}$$

$$\sigma^2(r_{d,t-1}) = E[(r_{d,t} - r_{d,t-1})^2|r_{d,t-1}] + u_t = E[(\Delta r_{d,t})^2|r_{d,t-1}] + u_t \tag{18}$$

where u_t are errors of $O(\Delta)$, and the correlation coefficient from

$$\rho = E\left[\frac{1}{\triangle\sigma_d(r_{d,t})\sigma_e(r_{e,t})} E_t\left[(r_{d,t+\triangle} - r_{d,t})(r_{e,t+\triangle} - r_{e,t})\right] + u_t\right] \tag{19}$$

Later in the article we compare the results for bond prices obtained from the above specifications with those obtained using the specifications $\mu_d(r_{d,t-1}) = E[\Delta r_{d,t}|r_{d,t-1}] + u_t$ and $\sigma_d^2(r_{d,t-1}) = E[(\Delta r_{d,t})^2|r_{d,t-1}] + u_t$, according to the diffusion

$$dr_d = \mu_d(r_d)dt + \sigma_d(r_d)dz_d \tag{20}$$

[8] We adopt now the more conventional notation of making the information set I_{t-1} and $\Delta r_{d,t} = r_{d,t} - r_{d,t-1}$. Therefore, we set $\Delta = 1$ and keep that notation for the rest of the paper.

We call this equation *Model 2*: it corresponds to the nonparametric specification of the basic one-factor process. By estimating this model and comparing the results obtained with those from *Model 1* we can assess the advantages of using the spread as the explanatory variable of the domestic interest rate dynamics.

In the next subsection we explain how the expectation terms in (12), (14) and (19) can be estimated nonparametrically and offer some comments on the estimation process.

2.2 Estimation of conditional moments with LLR

The simple Nadaraya-Watson (NW) estimator for the conditional expectation of a variable y given some conditioning variable x comes from defining

$$y = E[y|x] + u = m(x) + u \tag{21}$$

where $m(x)$ is a conditional expectation function and u is by construction an error term with zero mean conditional on x. The NW estimator calculates a local estimate $\widehat{m}(x_1)$ from the data by weighting the values of y around x_1 using a kernel function $k(x_i)$ that weighs the observations depending on their distance from x_1. The estimator is:

$$\widehat{E}\left[y|x_1\right] = \widehat{m}(x_1) = \frac{\frac{1}{N \cdot h_x} \sum_{i=1}^{N} k(x_i) \cdot y_i}{\frac{1}{N \cdot h_x} \sum_{i=1}^{N} k(x_i)} \tag{22}$$

where $\frac{1}{N \cdot h_x} \sum_{i=1}^{N} k(x_i) = \frac{1}{N \cdot h_x} \sum_{i=1}^{N} K(\frac{x_1 - x_i}{h_x}) = \widehat{f(x_1)}$ and h_x is a smoothing parameter that controls how distant observations are weighted into the local expectation.

The NW estimator can be thought of as a weighted least squares procedure performing a pointwise minimization of $\sum_{i=1}^{N} \{y_i - \alpha\}^2 k(x_i)$ with respect to α. If instead

we minimize

$$\sum_{i=1}^{N} \{y_i - \alpha - (x_i - x_1)\beta\}^2 k(x_i) \tag{23}$$

with respect to both α and β, then $\widehat{m}(x_1)$ can be identified with the estimate of α. Tthe resulting estimator has the form:

$$\widehat{m}(x_1) = \sum_{i=1}^{N} w_i(x_1) \cdot y_i \tag{24}$$

where the weights are $w_i(x_1) = e_1'(\sum_{i=1}^{N} z_i k_i z_i')^{-1} \cdot z_i k_i$, $k_i = K\left(\frac{x_1 - x_i}{h_x}\right)$, $z_i = \begin{pmatrix} 1 & (x_i - x_1) \end{pmatrix}'$ and $e_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is the vector that selects $\widehat{\alpha}$ from $\begin{pmatrix} \widehat{\alpha} & \widehat{\beta} \end{pmatrix}$. This local linear regression (LLR) estimator is a weighted least squares regression of y_i against z_i with weights set to $k_i^{1/2}$. The LLR is now fitting a straight line with nonzero slope[9] to the data close to x_1 and does this fitting locally for each possible value of x.

2.3 The Choice of Kernel and Window Width

The choice of the kernel function has been shown to be of relatively little importance or, at least, of less importance than that of the window width h_x. Most of the kernel functions that meet certain requirements yield results comparable to those of the theoretically optimal choice. Possible choices of the kernel function are the Triangular, Epanechnikov or Gaussian. We show the results of a Gaussian kernel: $K(\varphi_i) = (2\pi)^{-\frac{1}{2}} \cdot e^{-\frac{\varphi_i^2}{2}}$ although some other kernels –Epanechnikov and quartic– were tried with the results being unchanged.

The choice of the smoothing parameter h_x is more relevant. If the function is oversmoothed or undersmoothed (that is, if h_x is higher or smaller than some optimal

[9] Nadaraya-Watson could be thought of fitting a line with zero slope. Therefore LLR must do better, for at least locally it is correct to approximate smooth functions with a linear function.

value) the actual estimation might differ substantially from the true function $m(x)$. The optimal window width, calculated so that some error criterion (e.g. AIMSE, Asymptotic Integrated Mean Square Error, a polynomial approximation in h to the IMSE) is minimized, depends on the number of observations and the dispersion of those observations. Formal derivations of the AIMSE can be found in Härdle (1990), Scott (1992) and Pagan and Ullah (1999). Throughout the paper we use Silverman's rule of thumb ($h_i = 1.06 \cdot \widehat{\sigma}_i \cdot N^{\frac{-1}{d+4}}$) that is close to optimal when using a Gaussian kernel. A variable bandwidth estimator, as in Fan and Gijbels (1992), was also tried. This variable bandwidth estimator is controlling for the lower density of the conditioning variable in some areas of its range. The bandwidth is specified as being proportional to the density of the conditioning variable, $h_i \propto f(x_i)^{-1/5}$, so it allows the window width to be larger where the density of observations is lower, alleviating the empty space phenomenon. The results from this estimator differed only slightly from the regular LLR, and precisely in the areas of *higher* density of the conditioning variable. Thus, it offered no real improvement over the simpler LLR.

2.4 Data and results for interest rates

The short-term interest rates used are weekly observations of the Italian and Spanish interbank one-month middle rates and the middle interest rate for one-month deposits in Ecus. Thus, we identify the second factor with the Ecu/euro rate.[10]

[10] The Ecu was a basket of currencies from different countries, several of which were much weaker than an economy satisfying the Maastricht criteria. This argument led some people to suggest the DM rate as a better proxy for r_e. Nevertheless we think that, during the sample period, uncertainty about which countries would join EMU justifies the use of the Ecu. Additionally, the Maastricht requirement did not force interest rates to converge exactly to German rates -something which at times seemed quite infeasible for some smaller countries- but to the average of the three countries with lowest inflation. Thus, the convergence requirement was specified in terms of an average rate, which we believe can be better proxied by the Ecu rate.

Table 1 provides descriptive statistics for the interest rate data, and Figure 1 shows their time evolution. The European and Italian series start on April 8, 1988 and the Spanish series on September 2, 1988.[11] All series end in October 16, 1998 so there are 550 observations for Europe and Italy and 529 for Spain. The sample period corresponds to the years prior to the creation of EMU, the period during which the domestic countries were trying to bring their rates in line with the European rates.[12]

Figures 2 and 3 present some visual evidence on how our chosen conditioning values affect the evolution of the domestic rate in Spain (r_s).[13] Figure 2 plots Δr_s, conditioned on the spread $r_e - r_s$ and Figure 3 shows the absolute value of Δr_s conditioned on the level r_s. Figure 2 shows how negative values of Δr_s occur when the difference $r_e - r_s$ is big but these Δr_s become smaller as so does $r_e - r_s$: there seems to be, therefore, some evidence of (nonlinear) convergence. Figure 3 gives evidence of the level effect on volatility.

Processes for the domestic rates are estimated following (17) and (18) where the conditional expectation terms will be estimated as in (24). The European process is estimated with $\widehat{\mu}_e\left[r_{e,t-1}\right] = \widehat{E}[\Delta r_{e,t}|r_{e,t-1}]$ and $\widehat{\sigma}_e^2(r_{e,t-1}) = \widehat{E}[(\Delta r_{e,t})^2|r_{e,t-1}]$. The correlation coefficient is calculated as the unconditional expectation of the term in (19).

This approach to estimating the volatility, which corresponds to the expression directly derived from the approximation to the stochastic Taylor-series expansion of

[11] The starting dates have been determined by data availability on Datastream. Final dates were determined by the creation of EMU, when domestic interbank rates were discontinued.

[12] The convergence requirements were not specified until 1990, and were included in the Maastricht Treaty in 1992. However, countries had interest rate convergence as a macro objective some time before then. Restricting our sample to post-1991 data would not change the results significantly, though.

[13] A more comprehensive set of figures is available upon request.

the variance term, has one advantage with respect to the alternative $\widehat{\sigma}_d^2(r_{d,t-1}) = \widehat{var}[\Delta r_{d,t}|r_{d,t-1}]$, suggested by Stanton and used in his paper for the results in the tables: in order to estimate the alternative specification $var[\Delta r_{d,t}|r_{d,t-1}] = E[(\Delta r_{d,t} - E\Delta r_{d,t})^2 |r_{d,t-1}]$ we need to plug in an estimate of the drift $E[\Delta r_{d,t}|r_{d,t-1}]$ and then estimate $E[(\widehat{u}_{d,t})^2 |r_{d,t-1}]$ where $\widehat{u}_{d,t} = \Delta r_{d,t} - E\widehat{\Delta r}_{d,t}$ is the residual after estimating the drift. This implies that for values of r_d where the drift is not very accurately estimated (generally those values that have low density of $r_e - r_d$, in our case corresponding usually to the values in the higher range of r_d) we would be plugging in a very defective estimate, thus carrying over the error to the volatility. In particular, when $r_e - r_d$ has a low density, the estimated drift tends to be very close to the actual observed value of $\Delta r_{d,t}$, since there are only a few observations close to that point and consequently the weight assigned to other observations is small. Therefore the estimated residual $\widehat{u}_{d,t}$ will tend to be small, and when squared and plugged in the formula above the effect is amplified. This "empty space" phenomenon, more relevant in the case of the variance, is the reason why some of the papers that estimate conditional second moments find that the estimated function peaks at some point and then it starts to decrease.[14] "Leave-one-out" estimators try to solve for this feature, but they then create the opposite effect, since the estimated residual becomes too big. The use of the approximation (18) avoids part of this problem by not having to use an estimated residual. In other words, we would be avoiding a "carry-over" effect on the volatility of a poorly estimated drift. However, both the drift and the volatility will still be poorly estimated when the density of the conditioning variable is low. Otherwise,

[14] See Aït-Sahalia (1996 a), Stanton (1997) and our own results for Italy and Spain. A similar "empty-space" phenomenon led us to use only one conditioning variable in the volatility of the domestic rate.

and noting that we are using the squared value of $\Delta r_{d,t}$, the estimated variance will not be smooth enough (it will be highly sensitive to the actual values of $\Delta r_{d,t}$) and will yield a wiggly estimate for observations in the low density range of $r_{d,t}$.

Results of the drift and volatility (variance) functions for the European, Italian and Spanish rates are shown in Figures 4, 5 and 6.[15] We find nonlinear drifts and level-related heteroskedasticity in the diffusions, which suggests that the hypotheses underlying many parametric term structure characterizations are probably inaccurate.[16] We also show bootstrap confidence bands for the estimated functions.[17] The bootstrap bands have been calculated using 10000 replications of Künsch's (1988) block bootstrap algorithm with block length five (different block lengths yielded similar results).

The estimated correlation coefficients between dr_e and dr_s, and dr_e and dr_l are 0.24 and 0.21 respectively.

3 The market price of risk

In order to calculate the prices of assets that depend on the interest rate via simulation, we need to find expressions for the market price of interest rate risk, that will then allow us to simulate the risk-adjusted process for the interest rate. We first review the expression for the market price of European interest rate risk –which is

[15] As mentioned above, we tried some alternative estimators -higher order kernels and a variable bandwidth estimator (Fan and Gijbels 1992)- with the results not being significantly changed.

[16] In the case of Italy the drift is nonexistent. This result contrasts markedly with the simple OLS estimation that yields a significant mean reversion coefficient due to the effect of the outliers.

[17] We report the results obtained for the *bootstrap standard deviation* confidence intervals. *Traditional* and *bootstrap percentile* intervals are available to any interested reader. Traditional confidence bands are calculated from the asymptotic distribution of the LLR estimator (Pagan and Ullah, 1999) but they do not account for possible dependencies in the data. For this reason, in time series contexts researchers have opted for using bootstrap-based bands. For details on the two types of bootstrap bands mentioned see Efron and Tibshirani (1993).

already available in Stanton (1997)– and then derive the necessary expressions for estimation of the domestic price of interest rate risk.

3.1 European interest rate risk

The European short-term interest rate is the only factor affecting European zero coupon bonds. In this case the stochastic process followed by the spot interest rate is (4)

$$dr_e = \mu_e(r_e)dt + \sigma_e(r_e)dz_e \tag{25}$$

The price at time t of a zero coupon bond that depends on $r_{e,t}$ and matures at time T evolves according to:

$$\frac{dP(r_e,\tau)}{P} = \alpha_e(r_e,t)dt + \upsilon_e(r_e,t)dz_e \tag{26}$$

where $\alpha_e(.)$ is the expected instantaneous rate of return, $\upsilon_e^2(.)$ is its instantaneous variance and $\tau = T - t$.[18]

Since the drift and diffusion of r_e are time homogeneous, applying Ito's lemma yields the following expressions

$$\begin{aligned} \alpha_e(r_e,t)P &= \frac{1}{2}\sigma_e^2(r_e)P_{r_er_e} + \mu_e(r_e)P_{r_e} + P_t \\ \upsilon_e(r_e,t)P &= \sigma_e(r_e)P_{r_e} \end{aligned} \tag{27}$$

where $P_{r_er_e}$, P_{r_e} and P_t denote partial derivatives.

The no-arbitrage condition implies $\alpha_e(r_e,t) = r_e + \lambda_e(r_e)\frac{P_{r_e}}{P}$. We can substitute for α_e in the previous expression and set the drift equal to zero. The result is the following partial differential equation:

$$[\mu_e(r_e) - \lambda_e(r_e)]P_{r_e} + \frac{1}{2}\sigma_e^2(r_e)P_{r_er_e} + P_t - r_eP = 0 \tag{28}$$

[18]The price of the bond should be $P_t(T) = P(r_e,t,T)$. Since the drift and difussion of r_e do not depend on time the price of the bond becomes a function of r_e and maturity ($\tau = T - t$).

which is the bond pricing equation. The appropriate boundary condition is $P(r_e, 0) = 1$.

Note that this is equivalent to performing a change of measure in the original stochastic process since under the new measure the drift of the bond is zero but the original volatility does not change

$$\frac{dP(r_e, \tau)}{P} = \upsilon_e(r_e, t) dz_e^* \tag{29}$$

and the corresponding modified stochastic process for r_e is

$$dr_e = [\mu_e(r_e) - \lambda_e(r_e)]dt + \sigma_e(r_e) dz_e^* \tag{30}$$

In this case we can apply directly the approach of Stanton (1997), and calculate the market price of risk $\lambda_e(r_e)$[19] with the first order approximation,

$$\lambda_e(r_{e,t}) = \frac{\sigma_e(r_{e,t})}{\Delta(\sigma_e^{(1)}(r_{e,t}) - \sigma_e^{(2)}(r_{e,t}))} E_t(R_{t,1}^{(1)} - R_{t,1}^{(2)} | r_{e,t}) + O(\Delta) \tag{31}$$

where $R_{t,1}^{(1)}$, $R_{t,1}^{(2)}$ are the holding period returns between times t and $t+1$ on two nondividend paying securities dependent on the European rate. We have used the fact that $\lambda_e(r_{e,t})$ is related to the excess return on interest rate dependent securities. $\sigma_e^{(i)}(r_{e,t})$ is the instantaneous volatility of European asset i, conditional on the European rate.

3.2 Domestic interest rate risk

We proceed now to one of the main contributions of this paper. In this section we derive an expression that allows the nonparametric estimation of the market price of risk of the rate that depends on an additional source of uncertainty to be estimated

[19] This specification of λ satifies the conditions necessary to preclude the arbitrage opportunity pointed out by CIR (1985).

–the results in Stanton (1997) apply only to a one-factor process. The derivations here obtain the price of domestic interest rate risk, given the specification of our convergence model, although extensions to other settings could be similarly derived.

$P(r_d, r_e, \tau)$ is the price of a discount bond with face value one unit of domestic currency and τ periods to maturity. $P(r_d, r_e, \tau)$ depends on $r_d(r_d, r_e, t)$ (3); by Ito's lemma P must follow the SDE:[20]

$$\frac{dP(r_d, r_e, \tau)}{P} = \alpha_d(r_d, r_e, t)dt + \upsilon_d(r_d, t)dz_d + \upsilon_e(r_e, t)dz_e \tag{32}$$

with

$$\begin{aligned} \alpha_d(r_d, r_e, t)P &= \frac{1}{2}\sigma_e^2(r_e)P_{r_e r_e} + \frac{1}{2}\sigma_d^2(r_d)P_{r_d r_d} + \rho\sigma_e(r_e)\sigma_d(r_d)P_{r_e r_d} + \\ &\quad + P_t + \mu_d(r_e - r_d)P_{r_d} + \mu_e(r_e)P_{r_e} \\ \upsilon_e(r_e, t)P &= \sigma_e(r_e)P_{r_e} \\ \upsilon_d(r_d, t)P &= \sigma_d(r_d)P_{r_d} \end{aligned} \tag{33}$$

The two Wiener processes are correlated with coefficient ρ, and $P_{r_e r_e}$, $P_{r_d r_d}$, $P_{r_e r_d}$, P_{r_d}, P_{r_e} and P_t are partial derivatives.

In a two-factor economy the risk of each asset relative to each factor must be proportional to the sensitivity towards that factor. Thus, the no-arbitrage condition is $\alpha_d(r_d, r_e, t) = r_d + \lambda_d(r_d)\frac{P_{r_d}}{P} + \lambda_e(r_e)\frac{P_{r_e}}{P}$. We substitute for α_d and set the drift equal to zero to obtain the PDE for the bond

$$\begin{aligned} 0 &= \frac{1}{2}\sigma_e^2(r_e)P_{r_e r_e} + \frac{1}{2}\sigma_d^2(r_d)P_{r_d r_d} + \rho\sigma_e(r_e)\sigma_d(r_d)P_{r_e r_d} + P_t + \\ &\quad + [\mu_d(r_e - r_d) - \lambda_d(r_d)]P_{r_d} + [\mu_e(r_e) - \lambda_e(r_e)]P_{r_e} - r_d P \end{aligned} \tag{34}$$

[20] More exactly, the price of the bond should be $P_t(T) = P(r_d, r_e, t, T)$. Since the drift and difussion of r_d do not depend on time the price of the bond becomes a function of r_d, r_e and maturity ($\tau = T - t$).

The boundary condition for this PDE is $P(r_d, r_e, 0) = 1$.

If there are no arbitrage possibilities, the same probability will convert all asset prices into martingales (see Duffie, 2001). Under the new measure we have

$$dr_d = [\mu_d(r_e - r_d) - \lambda_d(r_d)]dt + \sigma_d(r_d)dz_d^*$$

$$dr_e = [\mu_e(r_e) - \lambda_e(r_e)]dt + \sigma_e(r_e)dz_e^* \tag{35}$$

To calculate the price of domestic interest rate risk we consider the function of the excess return of asset $P^{(1)}$ over $P^{(2)}$

$$f(r_d, r_e, s) = \frac{P^{(1)}(r_d, r_e, s)}{P^{(1)}(r_{d,t}, r_{e,t}, t)} - \frac{P^{(2)}(r_d, r_e, s)}{P^{(2)}(r_{d,t}, r_{e,t}, t)} \tag{36}$$

From the definition (6) of the infinitesimal generator, $\Im$ we get

$$\begin{aligned}
\Im f(r_{d,t}, r_{e,t}, t) &= \lim_{s \downarrow t} \frac{E_t[f(r_d, r_e, s)|r_{d,t}, r_{e,t}] - f(r_{d,t}, r_{e,t}, t)}{s - t} = \qquad (37)\\
&= \frac{1}{P^{(1)}(r_{d,t}, r_{e,t}, t)}[\frac{1}{2}\sigma_e^2(r_e)P^{(1)}_{r_e r_e} + \frac{1}{2}\sigma_d^2(r_d)P^{(1)}_{r_d r_d} + \\
&\quad + \rho\sigma_e(r_e)\sigma_d(r_d)P^{(1)}_{r_e r_d} + P^{(1)}_t + \mu_d(r_e - r_d)P^{(1)}_{r_d} + \mu_e(r_e)P^{(1)}_{r_e}] - \\
&\quad - \frac{1}{P^{(2)}(r_{d,t}, r_{e,t}, t)}[\frac{1}{2}\sigma_e^2(r_e)P^{(2)}_{r_e r_e} + \frac{1}{2}\sigma_d^2(r_d)P^{(2)}_{r_d r_d} + \\
&\quad + \rho\sigma_e(r_e)\sigma_d(r_d)P^{(2)}_{r_e r_d} + P^{(2)}_t + \mu_d(r_e - r_d)P^{(2)}_{r_d} + \mu_e(r_e)P^{(2)}_{r_e}]
\end{aligned}$$

We use (34) for

$$\begin{aligned}
\Im f(r_{d,t}, r_{e,t}, t) &= \frac{1}{P^{(1)}(r_{d,t}, r_{e,t}, t)}\left[\lambda_d(r_{d,t})P^{(1)}_{r_d} + \lambda_e(r_{e,t})P^{(1)}_{r_e}\right] - \qquad (38)\\
&\quad - \frac{1}{P^{(2)}(r_{d,t}, r_{e,t}, t)}\left[\lambda_d(r_{d,t})P^{(2)}_{r_d} + \lambda_e(r_{e,t})P^{(2)}_{r_e}\right]
\end{aligned}$$

or

$$\begin{aligned}
\Im f(r_{d,t}, r_{e,t}, t) &= \lambda_d(r_{d,t})\left[\frac{P^{(1)}_{r_d}}{P^{(1)}(r_{d,t}, r_{e,t}, t)} - \frac{P^{(2)}_{r_d}}{P^{(2)}(r_{d,t}, r_{e,t}, t)}\right] + \qquad (39)\\
&\quad + \lambda_e(r_{e,t})\left[\frac{P^{(1)}_{r_e}}{P^{(1)}(r_{d,t}, r_{e,t}, t)} - \frac{P^{(2)}_{r_e}}{P^{(2)}(r_{d,t}, r_{e,t}, t)}\right]
\end{aligned}$$

Now we use (33) to obtain

$$\Im f(r_{d,t}, r_{e,t}, t) = \lambda_d(r_{d,t}) \left[\frac{\sigma_d^{(1)}(r_{d,t}) - \sigma_d^{(2)}(r_{d,t})}{\sigma_d(r_{d,t})} \right] + \lambda_e(r_{e,t}) \left[\frac{\sigma_d^{(1)}(r_{e,t}) - \sigma_d^{(2)}(r_{e,t})}{\sigma_e(r_{e,t})} \right] \tag{40}$$

and substituting in equation (8)

$$\lambda_d(r_{d,t}) = \frac{\sigma_d(r_{d,t})}{\Delta(\sigma_d^{(1)}(r_{d,t}) - \sigma_d^{(2)}(r_{d,t}))} \left\{ E_t(R_{t,1}^{(1)} - R_{t,1}^{(2)} | r_{d,t}) - \lambda_e(r_{e,t}) \left[\frac{\sigma_d^{(1)}(r_{e,t}) - \sigma_d^{(2)}(r_{e,t})}{\sigma_e(r_{e,t})} \right] \right\} + O(\Delta) \tag{41}$$

where we already know the term $\lambda_e(r_{e,t})$ from the excess return of European assets. $\sigma_d^{(i)}(r_{e,t})$ is the volatility of domestic asset i conditional on the European interest rate, and $\sigma_d^{(i)}(r_{d,t})$ is the volatility of domestic asset i conditional on the domestic interest rate. $R_{t,1}^{(1)}$ and $R_{t,1}^{(2)}$ are again the returns of two nondividend paying securities dependent on the rate. Note that this formula also satisfies the conditions in CIR (1985).

3.3 Results for risk premiums

As proxies for the two nondividend paying securities we have used the one and six-month Italian and Spanish interbank middle rates. For the European case we have used the one and six-month Ecu denominated deposit rates. The use of Treasury Bills could have avoided some of the default risk inherent to interbank rates, but we did not have access to homogeneous T-Bill data for the three cases and decided to use a comparable rate. Even so, default risk in the three areas during the period studied was probably very small.

Results for the nonparametric estimation of the price of interest rate risk are shown in Figures 7 to 9. We observe a well known result in the Spanish case: the market price of risk is a decreasing function of the interest rate (Merton, 1990). In

the case of the European and Italian rates, the market price of risk seems to be close to zero (note the different scale in Figures 7 and 8). The average risk premiums are -0.00035 for Europe, and -0.00418 and 0.00365 for Spain and Italy. The European and Spanish risk premiums adopt the sign corresponding to a positive reward for carrying interest rate risk –this is what a priori one should expect– although the European risk is an order of magnitude smaller than the Spanish risk. In the Italian case the sign is reversed. The cause for this phenomenon may reside in the high volatility that Italian interest rates were subject to during the period under study.

Later in the paper we find that for valuation purposes interest rate risk premiums do not have a substantial influence in the performance of the model: the pricing errors obtained with the estimated market price of risk differ little from those obtained by setting $\lambda = 0$. The short-term nature of our data and the noise during the estimation period might be behind this result. It is not the first time that market risk premiums are not significant or even negative. In fact, ongoing research is still trying to explain this so-called "risk premium puzzle" (see Brennan and Schwartz, 1980, for an early motivation, and Gómez and Martínez, 2002, for a discussion on the market price of risk in Spain).[21]

We obtained an interesting result. If the European price of interest rate risk is set equal to zero the results for the estimation of the Spanish and Italian risk premiums –and their pricing effects– do not change: mean risk premiums are -0.00413 for Spain and 0.00368 for Italy. Therefore, during the period under study there seemed to be no premium for European interest rate risk. This is not unreasonable, given the low risk premium for both Spain and Italy and the fact that the European rate was composed

[21] This result has also been documented for Europe (Guo 1998 and Fiorentini et al. 2002).

mainly of economies more solid than those of Spain and Italy.

4 Pricing European and domestic bonds

4.1 European bonds

The European bond is affected solely by the European short-term interest rate so the price of a zero coupon bond with a payoff of \$1 at time T, $P_t(T)$, can be written as

$$P_t(r_e, T) = E_t^* \left[e^{-\int_t^T r_u^* du} \right] \tag{42}$$

where E_t^* denotes the expectation taken with respect to the modified stochastic process

$$\begin{aligned} r_e^* &= r_e \\ dr_e^* &= \{\mu_e(r_e^*) - \lambda_e(r_e^*)\} dt + \sigma_e(r_e^*) dz_e^*. \end{aligned} \tag{43}$$

We use Monte Carlo simulation to calculate the implied prices of bonds. This entails repeatedly simulating paths for the risk adjusted interest rate process r_e^* using the Euler discretization of the modified dynamics of (4) described in equations (43)

$$\triangle r_{e,t} = r_{e,t} - r_{e,t-\Delta t} = [\mu_e(r_{e,t-\Delta t}) - \lambda_e(r_{e,t-\Delta t})] \triangle t + \sigma_e(r_{e,t-\Delta t}) \sqrt{\triangle t} \xi_t \tag{44}$$

We simulated 10000 interest rate paths for each bond price. The ξ_t are drawn from a standard normal distribution.

4.2 Domestic bonds

In *Model 1* the domestic bond is affected by the domestic short-term interest rate whose drift depends on the spread with respect to the European rate and whose

volatility depends on its own level. Then the price of a zero coupon bond with a payoff of \$1 at time T, $P_t(T)$, can be written as

$$P_t(T) = E_t^* \left[e^{-\int_t^T r_u^* du} \right] \tag{45}$$

where E_t^* is the expectation taken with respect to the modified process

$$\begin{aligned} r_d^* &= r_d \\ r_e^* &= r_e \\ (r_e - r_d)^* &= (r_e - r_d) \\ dr_d^* &= \{\mu_d [(r_e - r_d)^*] - \lambda_d(r_d^*)\} dt + \sigma_d(r_d^*) dz_d^* \\ dr_e^* &= \{\mu_e(r_e^*) - \lambda_e(r_e^*)\} dt + \sigma_e(r_e^*) dz_e^* \end{aligned} \tag{46}$$

Calculations for equation (45) are done by simulating paths for the risk adjusted interest rate processes r_d^* and r_e^* using the following Euler discretization of the modified dynamics of (3) and (4)

$$\begin{aligned} \triangle r_{d,t} &= r_{d,t} - r_{d,t-\Delta t} = \{\mu_d [(r_e - r_d)_{t-\Delta t}] - \lambda_d(r_{d,t-\Delta t})\} \triangle t + \sigma_d(r_{d,t-\Delta t}) \sqrt{\triangle t} \xi_{d,t} \\ \triangle r_{e,t} &= r_{e,t} - r_{e,t-\Delta t} = \mu_e(r_{e,t-\Delta t}) \triangle t + \sigma_e(r_{e,t-\Delta t}) \sqrt{\triangle t} \xi_{e,t} \end{aligned} \tag{47}$$

calculating the integral inside the expectation in equation (45) for each path, and averaging over the paths. We simulated 10000 interest rate paths for each bond price. The ξ_t are drawn from a bivariate normal distribution with correlation coefficient ρ. The value for ρ comes from the nonparametric estimation of (16).

In the case of *Model 2* the equations are parallel to (43) and (44).

4.3 Results of bond pricing

4.3.1 The convergence model

We carry out the above simulations for one and two year bonds and compare, using the root mean squared error, RMSE, the results with actual bond prices. Weekly prices of discount bonds are obtained from the estimation of the yield curve fitted with splines to the power of three with maturities one and two years available in Datastream. We have a total of 382 prices of bonds in all three cases, corresponding to the period 9/90 to 12/97.

Results on the prices and pricing errors appear in Tables 2 and 3 for European and domestic bonds respectively. Differences in the results with and without the risk premiums are small, although the accuracy is slightly improved when the risk premium is accounted for. These results are a consequence of the fact that, on average, the risk premiums are very close to zero, especially in the case of the Italian and European bonds.

The comparison between the results of *Model 1* (convergence) and *Model 2* (Stanton 1997) uncovers a key feature. Pricing errors of *Model* 1 are 5.8% smaller for the one year Spanish bonds and 4.2% for the one year Italian bonds. The reduction of pricing errors is only 1.5 % for two year bonds. Given the increased flexibility provided by the second factor, the performance of the two-factor model improves for the short-term end of the term structure but this improvement vanishes as longer term bonds are priced. We comment on this result in the next subsection.

Regarding the nonparametric estimation, in Table 4 we compare our errors for the Spanish bonds with those obtained in the parametric estimation in CS (2000).[22]

[22] We do not have available studies to perform comparisons for the European and Italian cases.

Our errors are significantly smaller than those of both the parametric convergence and Vasicek models: for one and two year bonds we reduce the RMSE by 34%.[23]

4.3.2 Some comments on two-factor models

Given that the term structure depends on both the volatility and the level of the interest rate, incorporating one of the two as a second factor has been recognized as necessary to give a better account for the behavior of the term structure: two-factor models for interest rates have therefore included a stochastic-mean factor (BDF 1998; BDFS 2000) –of which our convergence model is a special case– or a stochastic-volatility factor (Andersen and Lund 1997; Downing 1999; BDFS 2000; Bali 2000, 2003; Boscher et al. 2000; Boudoukh et al. 2000).

Our analysis gives guidance in the specification of two-factor interest rate models. The differences in prices obtained with the one and two-factor models are small: *Model 1* outperforms *Model 2* in all instances, but more significantly for short-term bonds. The main feature of our convergence model is that, based on intuition coming from the economic behavior of the data, it incorporates an observable stochastic mean. We use a level effect for the variance (CKLS 1992; Pagan et al. 1996; Brenner et al. 1996), a choice that allows us to account for volatility without introducing a third factor.[24] This stochastic mean model performs quite well at the short end

Nevertheless, pricing errors are similar in magnitude to those of Spain.

[23] The time period and data used in CS (2000) vary slightly from those used here but the results in Table 4 incorporate the necessary adjustments.

[24] A GARCH variance or a stochastic volatility –and therefore a three-factor model– could have been used alternatively. The parametric literature agrees that the three specifications do a similar job when fitting the volatility of interest rates, although the stochastic volatility is slightly more flexible (Pagan 1996). The reason rests on the strong persistence of the volatility: Estimated volatilites present an autocorrelation function close to that of an integrated variable. Given that the interest rate itself is near-integrated, the autocorrelations of volatility implied by a volatility in levels term, by a GARCH model with GARCH parameter close to one –as it is frequently the case– and by a stochastic volatility model with autoregressive parameter close to one are all alike, showing a slow

of the term structure but the improved accuracy is lost for longer-term assets. The results of the pricing exercise therefore suggest that when the main objective is the correct valuation of short-term assets, a flexible specification of the behavior of the mean process –including a stochastic mean as the second factor– can substantially reduce the pricing errors, as long as the volatility incorporates a structure that allows for persistence and conditional heteroskedasticity. However, when longer-term assets are the focus of the analysis, allowing for stochastic volatility is probably the better way to obtain low pricing errors since in longer horizons the effect of the volatility overwhelms the effect of the weak mean reversion (see also Bali 2003). Of course, if one wants to fit the complete term-structure, a three-factor model would dominate, but the literature so far has favored keeping simpler models rather than running the risk of overfitting. A formal comparison of different two-factor models is beyond the scope of this paper, since our main concern was to present and justify the convergence model. However, analysis of the performance of alternative models at the different ends of the term structure provides fruitful and exciting avenues for future research.

Finally, allowing for nonlinearities in the behavior of the factors, which we have done through the nonparametric estimation, is shown to lead to much improved results. Thus, the derivations in our paper, that extend Stanton's (1997) methodology to a specific two-factor structure, or in Downing (1999), that also extends nonparametric estimation to multifactor models, are helpful when deciding on the correct specification and analysis of interest rate processes.

decrease similar to that of a near-unit root process.

5 Conclusions

In this paper we analyzed the dynamic behavior of the short-term interest rates of Italy and Spain as two-factor mean reverting processes: we modeled the convergence between interest rates –experienced in EMU countries– using the spread between the short term domestic and European interest rates as the underlying forcing variable. We used nonparametric techniques to estimate the short term interest rate dynamics. Important extensions to the results available in the literature were derived that allowed us to account for all the features of the convergence model. We then proceeded to price zero coupon bonds and to compare the results with actual market prices.

Estimation of the interest rate processes showed evidence in favor of the convergence model, yielding a nonlinear mean reverting behavior of domestic rates towards the European rate. With regards to bond prices, the performance of the convergence model is about a 5% better for one year bonds and a 2% better for two year bonds than a simpler one-factor model without the convergence feature. Additionally, pricing errors were about 34% smaller than those obtained with parametric models. A substantial improvement in the results in this paper, therefore, rests on the nonparametric technique used in the estimation. However, the second factor introduced in the drift, the main feature of the convergence model, does improve significantly the accuracy of the pricing of short duration bonds.

The above results provide directions for future research, especially with regards to the specification of multifactor models: models for the pricing of shorter-term interest rate derivatives seem to profit from incorporating more structure in the mean of the rate –thus suggesting the use of a stochastic mean factor– whereas the accuracy in

the pricing of longer-term derivatives seems to depend more on the specified behavior of volatility –suggesting the use of a stochastic volatility factor.

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Table 1

Descriptive Statistics. Weekly data

Period of study: 4/8/88 to 10/16/98 for the European and Italian rates; 9/2/88 to 10/16/98 for Spain. μ, σ, S and κ are, respectively, the mean, standard deviation, skewness and kurtosis coefficients. r_e is the European rate whereas r_l and r_s are, respectively, the Italian and Spanish rates.

	μ	σ	S	κ	*Max*	*Min*
Δr_e	-0.000038	0.002	-0.0349	16.560	0.014375	-0.015
r_e	0.07332	0.0247	0.0187	1.4699	0.126875	0.04
Δr_l	-0.000112	0.0059	3.0504	110.58	0.086	-0.0644
r_l	0.10304	0.0276	0.5314	5.679	0.2769	0.047825
r_e-r_l	-0.0301	0.0146	-1.7777	14.699	0.001875	-0.161587
Δr_s	-0.00012	0.00344	0.1066	31.539	0.0307	-0.0272
r_s	0.10475	0.0369	-0.1382	1.6555	0.1775	0.0415
r_e-r_s	-0.03143	0.0164	-0.1664	2.4127	-0.000875	-0.080625

Table 2

European Bonds. RMSE and Effect of Risk on Bond Valuation

Period of study: 9/90 to 12/97

Results of bond valuation for bonds maturing in one and two years. The Root Mean Square Error is calculated with the difference between the bond prices obtained with the Monte Carlo simulation and the actual market bond prices. The value of the bond corresponds to the average price of the results of the simulation.

1 Year	*RMSE*	0.00559
	Value of Bond	0.93422
2 Years	*RMSE*	0.01477
	Value of Bond	0.87329

Table 3

Domestic Bonds. RMSE and Effect of Risk on Bond Valuation

Period of study: 9/90 to 12/97

Results of bond valuation for bonds maturing in one and two years. The Root Mean Square Error is calculated with the difference between the bond prices obtained with the Monte Carlo simulation and the actual market bond prices. The value of the bond corresponds to the average price of the results of the simulation. Values in the first column use the estimated $\lambda(r_s)$. Values in the second column are calculated imposing both $\lambda(r_s) = 0$ and $\lambda(r_e) = 0$.

		Model 1		*Model 2*	
		$\lambda(r)$	$\lambda = 0$	$\lambda(r)$	$\lambda = 0$
		Spanish Bonds			
1 Year	*RMSE*	0.00631	0.0064	0.0067	0.00671
	Value of Bond	0.90788	0.90797	0.90799	0.90796
2 Years	*RMSE*	0.0143	0.01475	0.01485	0.01491
	Value of Bond	0.8257	0.8256	0.82545	0.82537
		Italian Bonds			
1 Year	*RMSE*	0.0078	0.0079	0.00815	0.00815
	Value of Bond	0.9041	0.9042	0.90434	0.90433
2 Years	*RMSE*	0.0159	0.0159	0.01619	0.01621
	Value of Bond	0.8177	0.8178	0.81864	0.81858

Table 4

Comparison of Parametric and Nonparametric results for the Spanish data

Period of study: 9/90 to 12/97.

Results of bond valuation for bonds maturing in one and two years. The Table shows the RMSE obtained with three different models: The nonparametric models estimated in this paper and the convergence and Vasicek models estimated in CS (2000). The Root Mean Square Errors are calculated with the difference between the bond prices estimated with the different techniques and the actual market bond prices.

	Model 1	*Model 2*	*Convergence*	*Vasicek*
1 Year	0.0063	0.0067	0.0095	0.0093
2 Years	0.0143	0.0148	0.0215	0.022

Figure 1

Short-Term Interest Rate Evolution

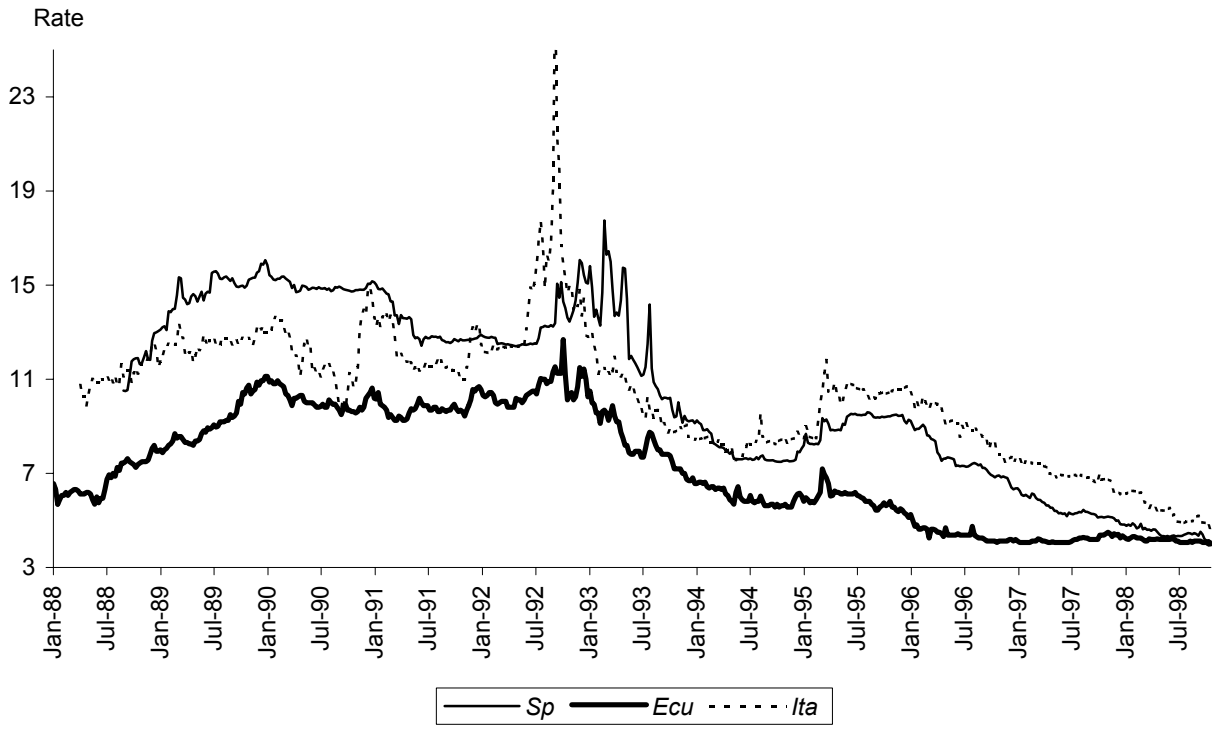


Figure 2

Changes in Spanish rate (drs) conditioned on differential with European rate (re-rs)

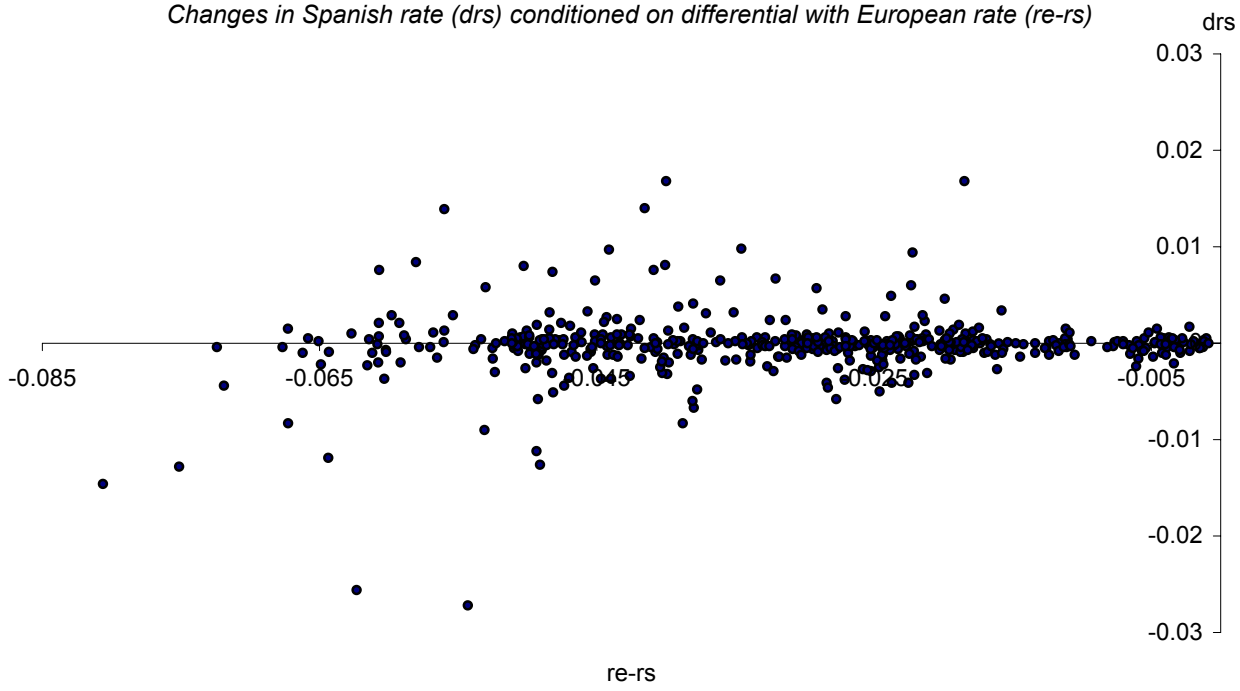


Figure 3

Absolute value of Changes in Spanish rate (drs) conditioned on its own level (rs)

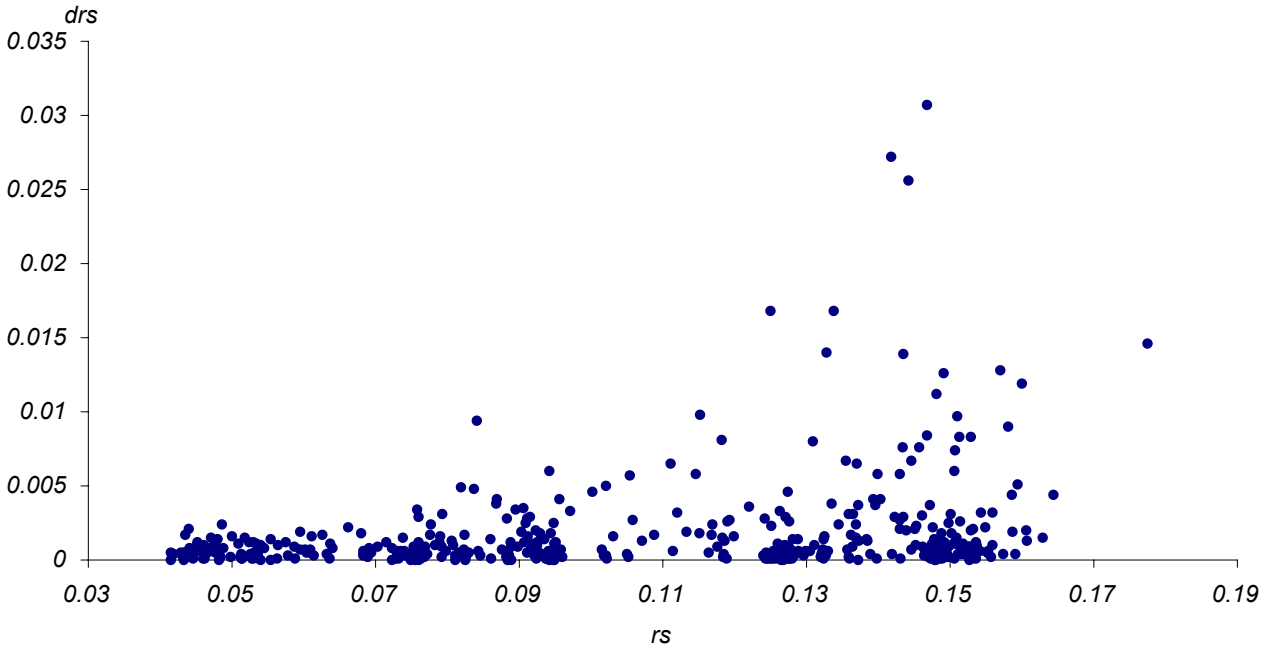
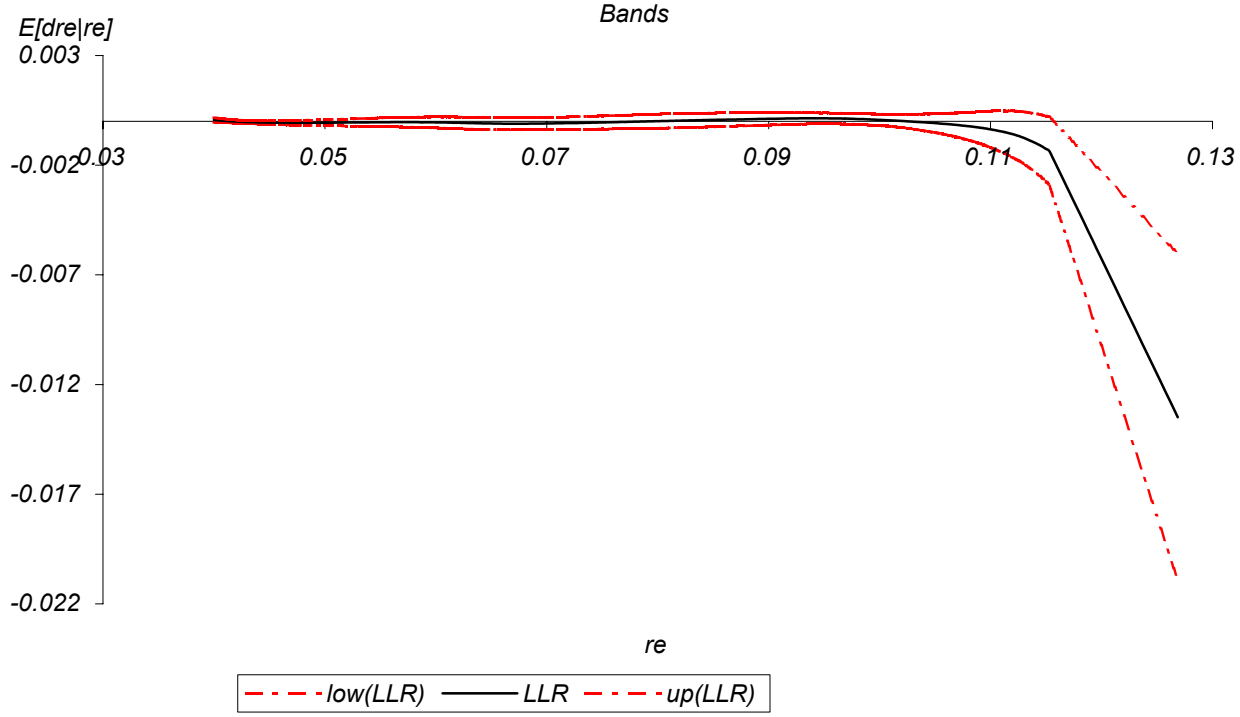


Figure 4

Drift of the European Short-term Interest Rate Estimated with LLR and LLRBV and Confidence Bands



Variance of the European Short-term Rate Estimated with LLR and Confidence Bands

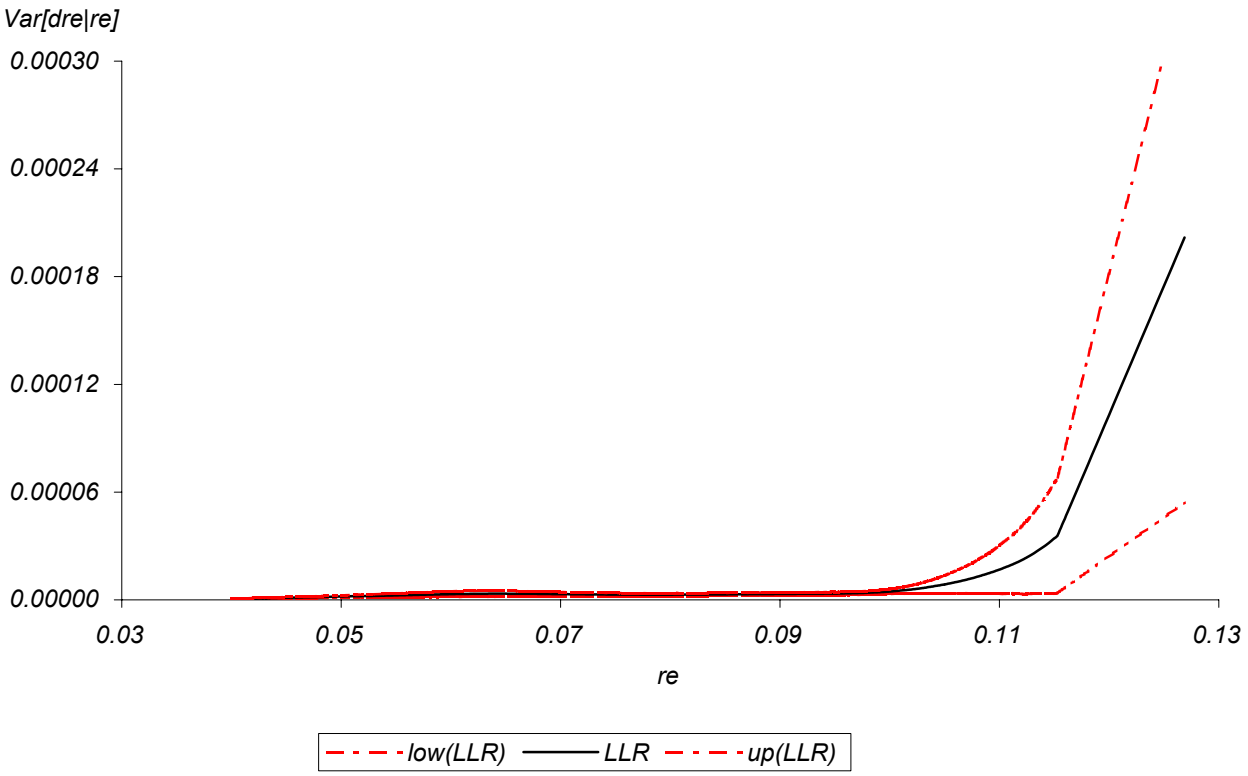
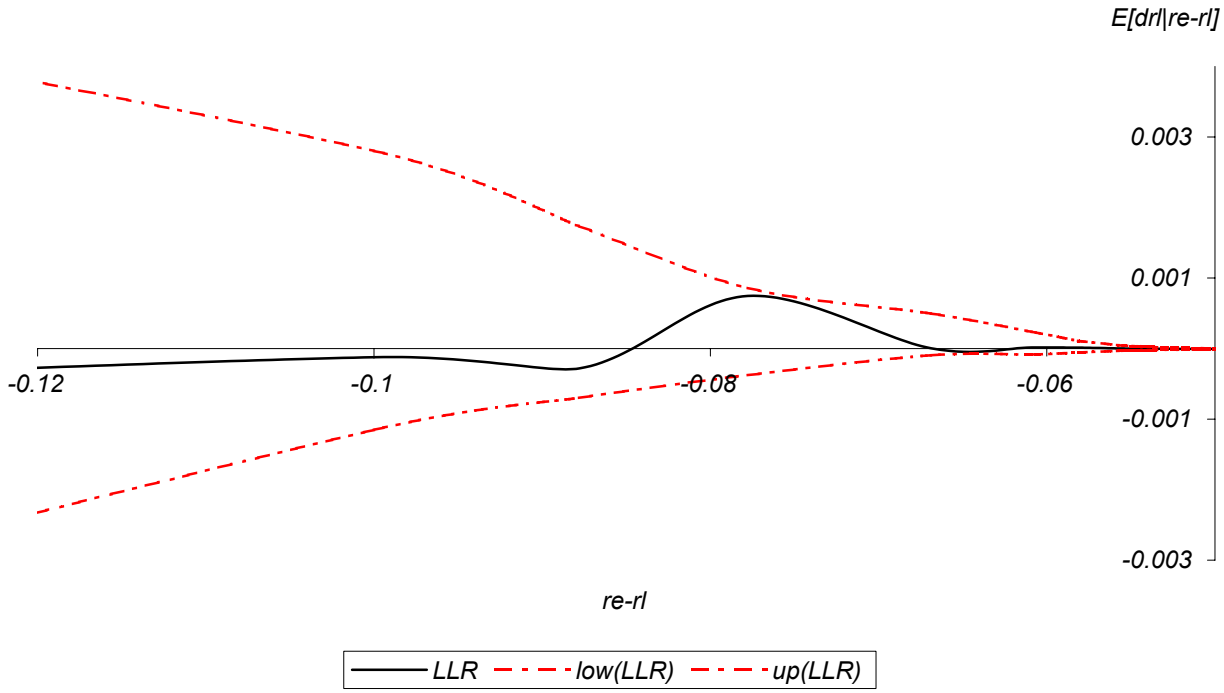


Figure 5

Drift of the Italian Short-term Interest Rate Estimated with LLR and Confidence Bands



Variance of the Italian Short-term of Interest Rates Estimated with LLR and Confidence Bands

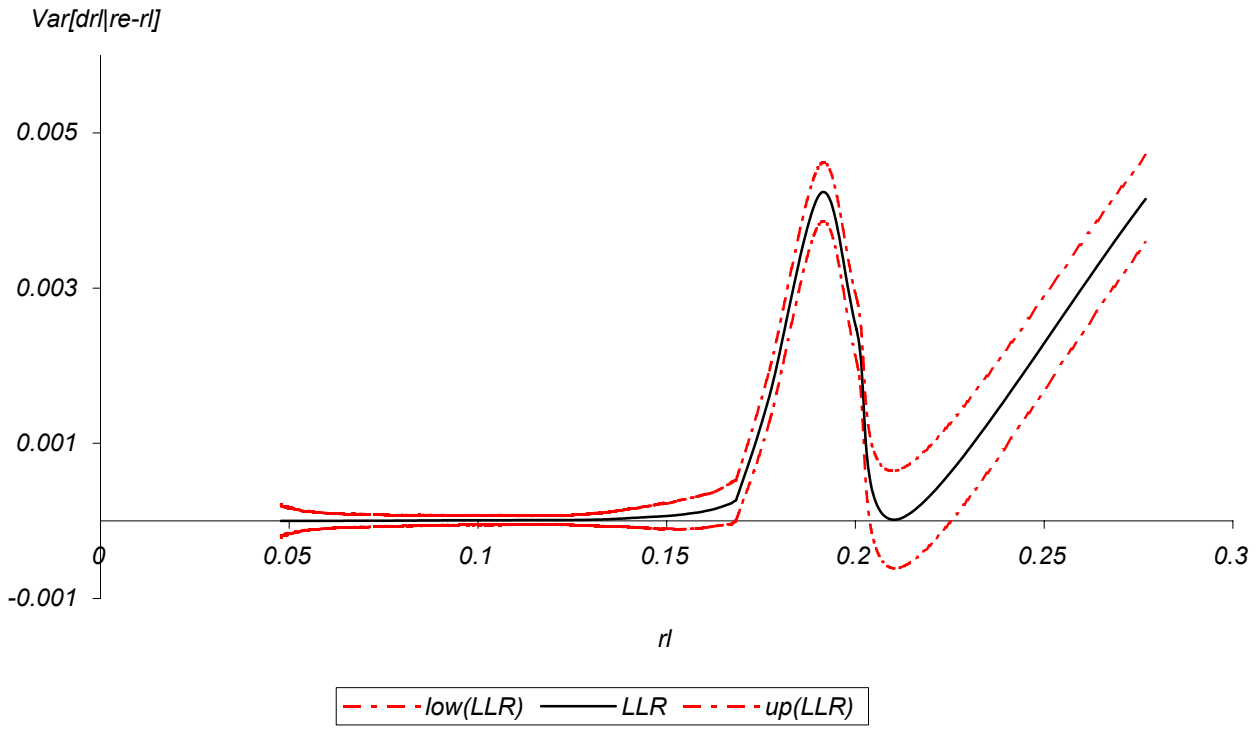
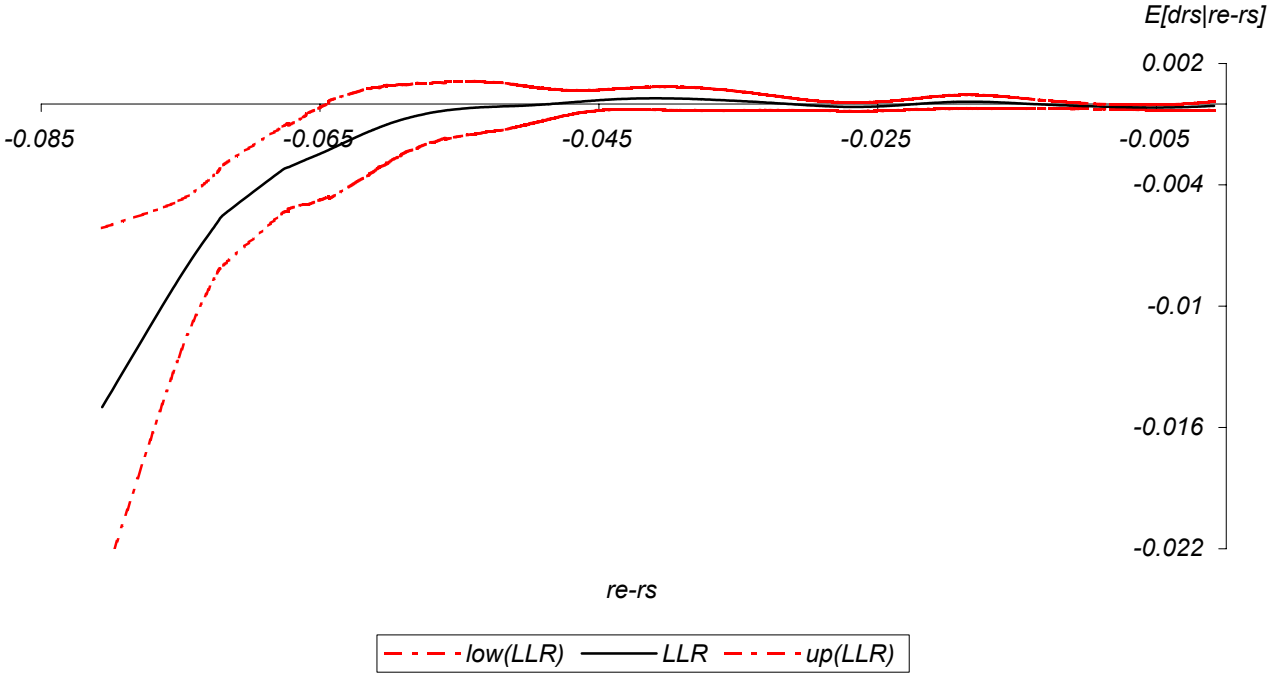
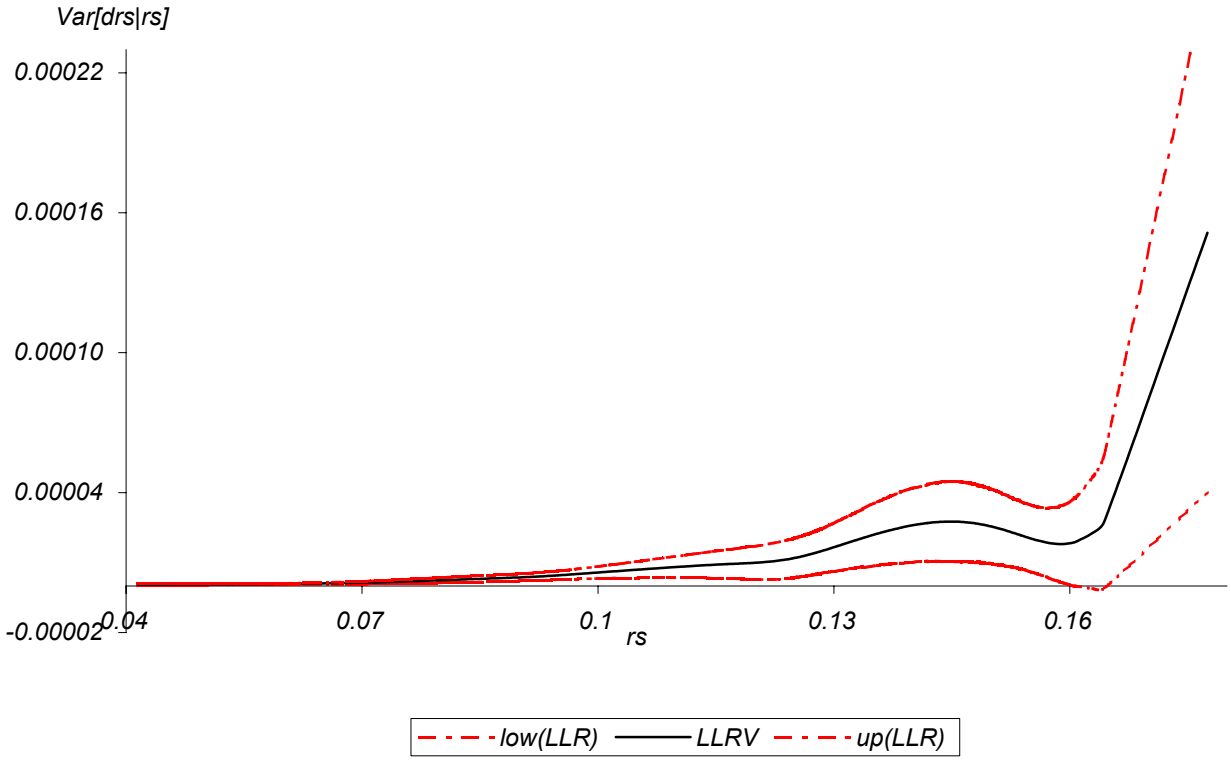


Figure 6

Drift of the Spanish Short-Term of Interest Rates Estimated with LLR and Confidence Bands



Variance of the Spanish Short-term Interest Rate Estimated with LLR and Confidence Bands



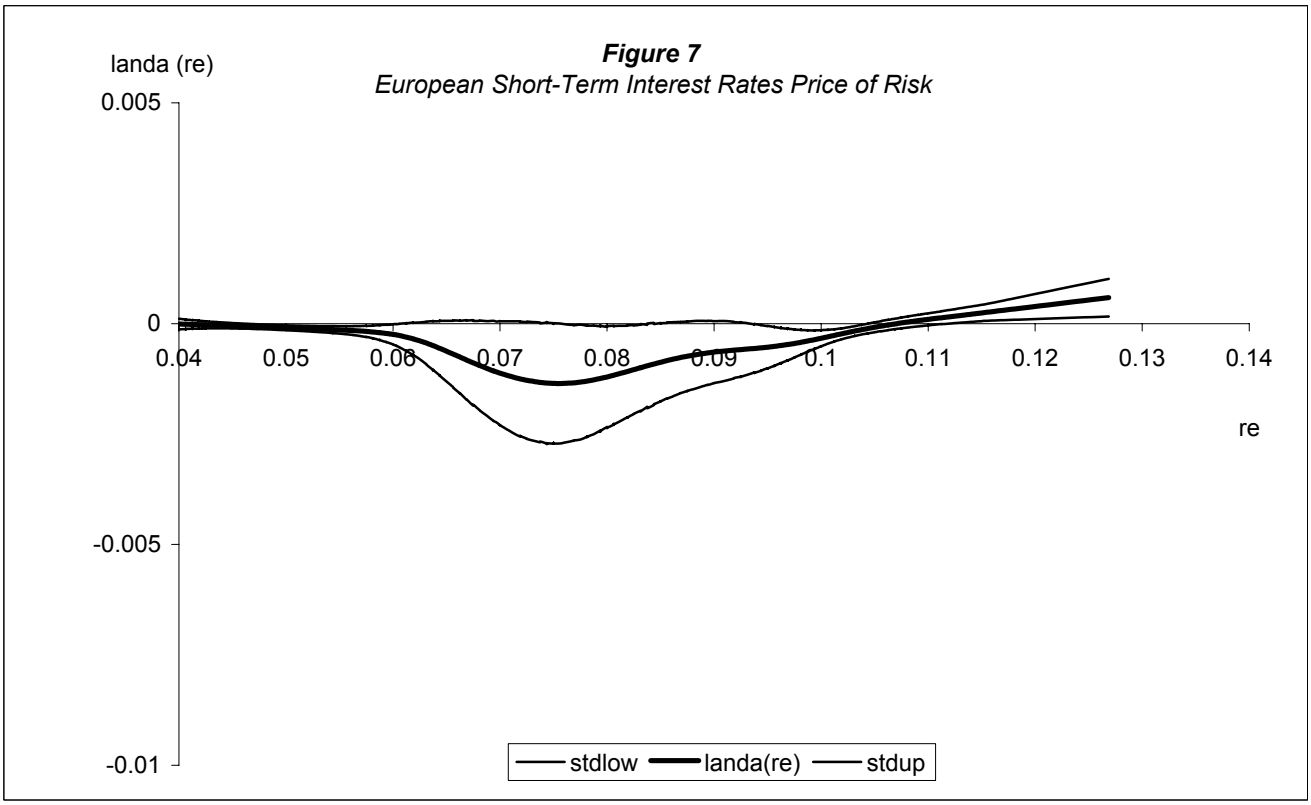


Figure 7
European Short-Term Interest Rates Price of Risk

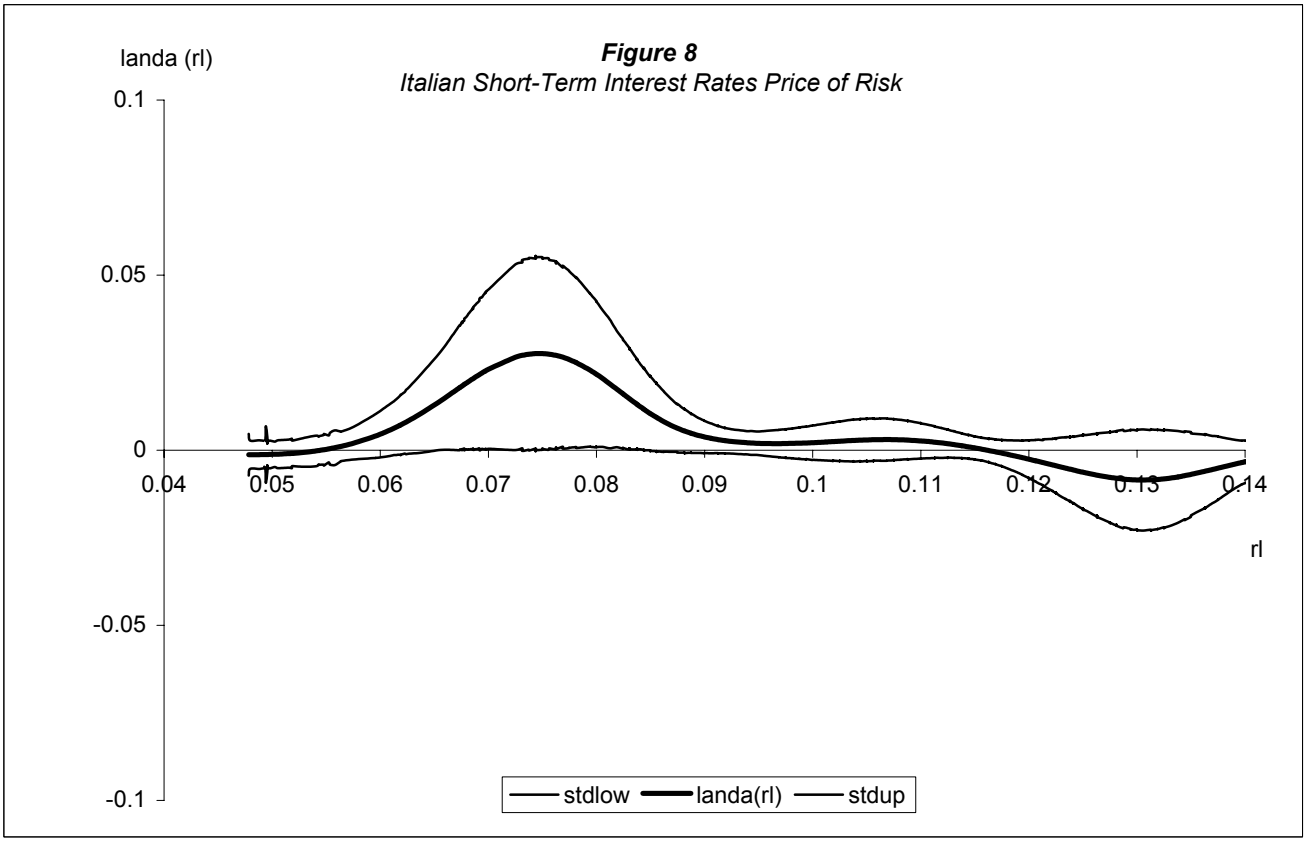


Figure 8
Italian Short-Term Interest Rates Price of Risk

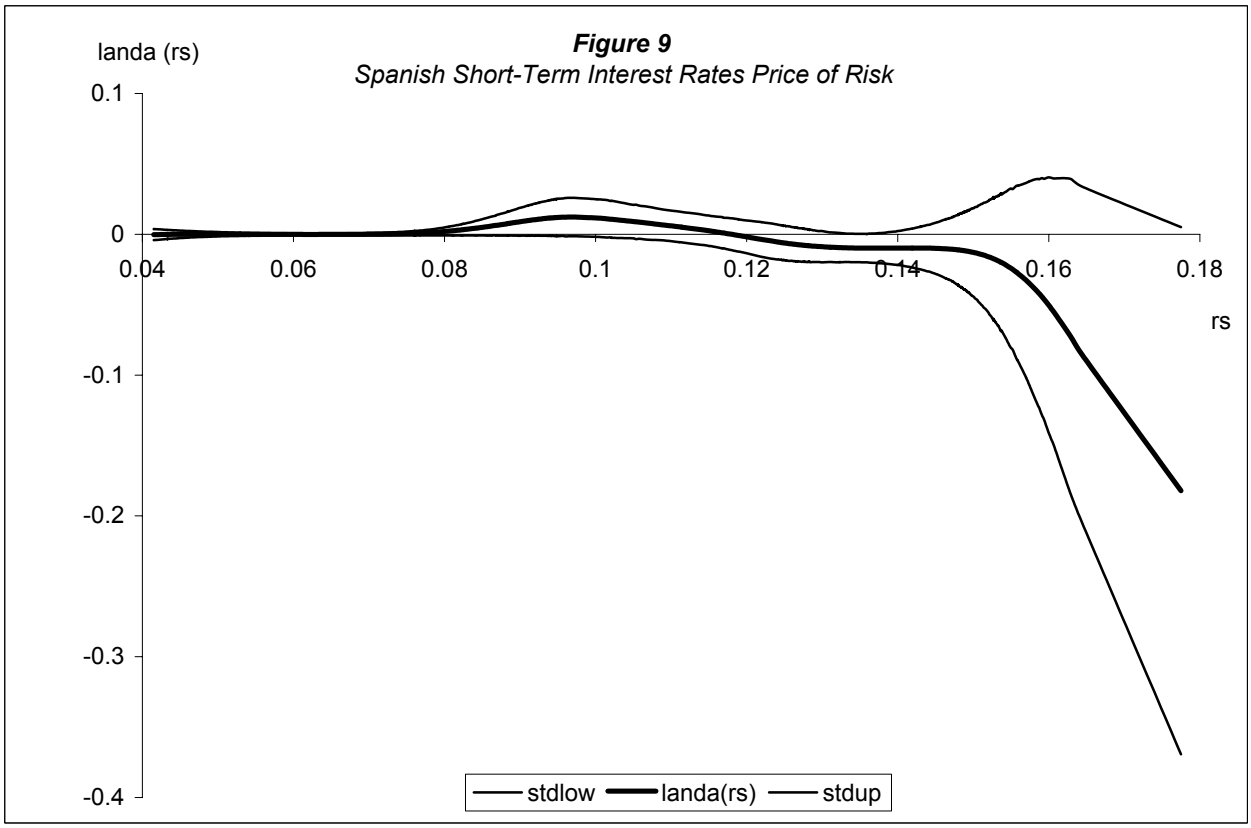


Figure 9

Spanish Short-Term Interest Rates Price of Risk