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ABSTRACT

Imperfect private monitoring in an infinitely repeated discounted Prisoner's Dilemma played on a communication network is studied. Players observe their direct neighbors' behavior only, but communicate strategically the repeated game's history throughout the network. The delay in receiving this information requires the players to be more patient to sustain the same level of cooperation as in a complete network, although a Folk Theorem obtains when the players are patient enough. All equilibria under exogenously imposed truthtelling extend to strategic communication, and additional ones arise due to richer communication. There are equilibria in which a player lies. The flow of information is related with network centrality measures.

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Abstract

Imperfect private monitoring in an infinitely repeated discounted Prisoner's Dilemma played on a communication network is studied. Players observe their direct neighbors' behavior only, but communicate strategically the repeated game's history throughout the network. The delay in receiving this information requires the players to be more patient to sustain the same level of cooperation as in a complete network, although a Folk Theorem obtains when the players are patient enough. All equilibria under exogenously imposed truthtelling extend to strategic communication, and additional ones arise due to richer communication. There are equilibria in which a player lies. The flow of information is related with network centrality measures.

JEL classification numbers: C72, C73, D85

Keywords: Repeated Game, Prisoner's Dilemma, Imperfect Private Monitoring, Network, Strategic Communication, Centrality

1 Introduction

The Prisoner's Dilemma is a well studied game, not only in Economics, since it captures many features from reality. The players' selfish behavior leads them to play the Nash

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Equilibrium in strictly dominant actions. However, this is not efficient and all players benefit from cooperation. This is achieved, and thus efficiency attained, by repeating the Prisoner's Dilemma forever, provided the players are patient enough.

This paper analyzes an infinitely repeated discounted Prisoner's Dilemma played on a network. Kinateder (2008) defines repeated network games based on any stage game. All players in a connected and undirected network that is fixed throughout the repeated game participate in the same stage game at each point in time. A player only observes his neighbors' behavior. However, by communicating with them, he receives the entire history of the repeated game with a finite delay. For patient players, a Folk Theorem obtains provided that they truthfully communicate their observations to their neighbors.

For the Prisoner's Dilemma additional results obtain, in particular, under strategic communication. All belief-free equilibria extend from exogenously imposed truthtelling to strategic communication and others exist due to richer communication. The players may lie, even in equilibrium, and imperfect private monitoring arises endogenously in this model. The information a player receives is a strategic choice of the other players.

In the literature, frequently, each player in a repeated Prisoner's Dilemma receives a distinct, exogenously determined and imperfect signal of each action profile played.[1] In the setup studied here, each player observes his neighbors perfectly, although the information they communicate him may contain lies. Hence, a slightly simpler version of belief-free equilibrium is used than in other imperfect monitoring models in which a player observes the repeated game's history with a vanishing $\varepsilon-$noise.

In case the players follow the trigger strategy profile, it is possible to relax the assumption of the network's connectedness and cooperation obtains if each group contains at least two players and they are patient enough.

The repeated network Prisoner's Dilemma is defined next. In section 3, the basic difference between the complete and a star network each formed by three players which follow the trigger strategy is illustrated.[2] This result is extended to any connected network, and conditions are given under which it holds for unconnected networks. In section 4, results under strategic communication are given. Before concluding, the results are allocated to the literature and the informational setup in the network studied here is related to network centrality measures.

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[1] A Folk Theorem obtains in Bhaskar and Obara (2002), Sekiguchi (1997), Ely and Välimäki (2002) and Piccione (2002). Kandori (2002) surveys the imperfect private monitoring literature. Fudenberg, Levine and Maskin (1994) obtain a Folk Theorem with imperfect public monitoring for any stage game.

[2] This part of the paper coincides with the Prisoner's Dilemma example given in Kinateder (2008).

2 Preliminaries

2.1 Prisoner's Dilemma Stage Game and Network

Each player i in the finite set of players $I = \{1, ..., n\}$, where $n > 2$, has a set of pure actions $A_i = \{C, D\}$; C stands for *cooperate* and D for *defect*. The stage game's pure action space is $A = \times_{i \in I} A_i$, with generic element a, called pure action profile. To emphasize player i's role, a is written as (a_i, a_{-i}). For any subset of players $S \subset I$, let $A_S = \times_{i \in S} A_i$, and denote by a_S an element of this set. Player i's payoff function is a mapping $h_i : A \to \mathbb{R}$, and the payoff function $h : A \to \mathbb{R}^n$ assigns a payoff vector to each pure action profile. Given $a \in A$, player i's payoff function is

$$
h_i(a) = \begin{cases}
3 & \text{if } a_j = C \text{ for all } j \in I \\
0 & \text{if } a_i = C \text{ and } \exists\, j \in I \setminus \{i\} \text{ s.t. } a_j = D \\
4 & \text{if } a_i = D \text{ and } a_j = C \text{ for all } j \in I \setminus \{i\} \\
2 & \text{if } a_i = D,\ \exists\, j \in I \setminus \{i\} \text{ s.t. } a_j = D \text{ and } \exists\, l \in I \setminus \{i, j\} \text{ s.t. } a_l = C \\
1 & \text{if } a_j = D \text{ for all } j \in I.
\end{cases}
$$

A player's payoff is 3 when all players choose C. It is 4 if he chooses D unilaterally. It is 2 if some other player chooses D as well while at least one player chooses C. It is 0 if he chooses C while at least one other player chooses D and it is 1 if all players choose D.

The Prisoner's Dilemma stage game in normal form is the tuple $G \equiv (I, (A_i)_{i \in I}, (h_i)_{i \in I})$. Let the convex hull of the finite set of payoff vectors corresponding to pure action profiles in G be $co(G) = co\{x \in \mathbb{R}^n \mid \exists\, a \in A : h(a) = x\}$.

The players in set I are vertices of a network g, whose graph is defined as (I, E), where $E \subseteq I \times I$ denotes the set of links between them. A link from player i to player j is denoted by (i, j). Graph (I, E) is undirected, that is, for all $i, j \in I$, $(i, j) \in E$ if, and only if, $(j, i) \in E$. Given network g, a path between two distinct players i and j is defined as a sequence of distinct players $i_1, ..., i_r$ with $i_1 = i$, $i_r = j$, and $(i_{l-1}, i_l) \in E$, for all $1 < l \leq r$. Its length is $r - 1$. Let network g be connected, that is, each player is connected to at least one other player directly and to all others via paths of finite lengths. The length of the shortest path between two distinct players i and j is called *distance* between i and j. It is denoted by d_{ij}. The *largest distance* (along shortest paths) between player i and any other player in g is defined by $d_i = \max_{j \in I} d_{ij}$, and network g's *diameter* is the maximal *largest distance* among all players, that is, $d = \max_{i \in I} d_i$. Finally, denote player i's set of direct neighbors by $i(1) = \{j \in I \mid d_{ij} = 1\}$, and for any $2 \leq m \leq d_i$, define his set of *m-neighbors* as $i(m) = \{j \in I \mid d_{ij} = m\}$.

2.2   Communication and Observations

When the Prisoner's Dilemma is played repeatedly, in each period, a player first chooses an action, in a way specified below, and then makes observations and communicates with his neighbors. At any $t \geq 1$, let player i's *set of observations* be Ob_i^t. It includes all possible observations that i may make at t of the actions chosen by his direct neighbors and the information they observed one period earlier which they communicate him strategically. At any $t \geq 1$, let player i's observation be $ob_i^t \in Ob_i^t$, and denote the observation profile by $ob^t \in Ob^t$, where $Ob^t = \times_{i \in I} Ob_i^t$.

At any $t > 1$, player i sends a report r_i^t from his set of reports R_i^t, to be defined later, to all neighbors in $i(1)$. He reports the information he received at $t-1$ in a strategic way, that is, possibly lying. Given ob_i^{t-1}, player i *lies* at t if his report r_i^t differs from ob_i^{t-1} as follows: he changes the action $a_j^s \in ob_i^{t-1}$, abusing notation, chosen by some player j at some $1 \leq s < t$ to any other action $b_j \in A_j \setminus \{a_j\}$. He reports all other observations in ob_i^{t-1} truthfully to his neighbors. Apart from his report, i's neighbors also observe his action choice at t.

Observations and reports evolve as follows. At $t = 1$, $Ob_i^1 = A_i \times A_{i(1)}$, that is, player i observes what he and all his neighbors do, while $R_i^1 = \emptyset$ since i has no previous information to report. At the end of $t = 2$, player i reports to any neighbor in $i(1)$ what he observed that all his neighbors did at $t = 1$. Formally, $R_i^2 = A_{i(1)} = Ob_i^1 \setminus A_i$ since a player never reports what he did, unless this is part of a report he received from a neighbor. At $t = 2$, player i observes what his neighbors do and receives their reports, that is, $Ob_i^2 = A_i \times A_{i(1)} \times_{j \in i(1)} R_j^2$. In this way, a recursive dynamic process of observations and reports is generated, and at any $t > 1$, $R_i^t = Ob_i^{t-1} \setminus A_i$ and $Ob_i^t = A_i \times A_{i(1)} \times_{j \in i(1)} R_j^t$.[3]

Information flows one link per period, and at $t = d_i$, player i for the first time receives a filtered version of what the most distant player from him did at $t = 1$. It is filtered by the players located on the shortest path between him and the player at distance d_i from him. However, links are not only counted along shortest paths, but a link is used several times on any path (for example, any piece of information flows back and forth between two neighbors). Hence, a player's observations and reports grow in size over time since he receives one report from any neighbor and hands it over to all neighbors (including the one from which he received it) in the subsequent period.

The players have perfect recall, and player i's set of private histories at the end of

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[3]At $t = 3$, player i reports to every $j \in i(1)$ what he observed them doing at $t = 2$ and what they told him at $t = 2$, which includes what all players 2 links away from i did at $t = 1$. Since player i is two links away from himself, at $t = 3$, he tells every neighbor what they told him that he did at $t = 1$.

period t is denoted by $\mathcal{H}_i^t = \cup_{s=1}^t Ob_i^s$. The private history he observed at the end of t is thus $\cup_{s=1}^t ob_i^s$. The players organized in this way play an infinitely repeated discounted Prisoner's Dilemma.

2.3 Repeated Prisoner's Dilemma Played on a Network

In the infinitely repeated discounted Prisoner's Dilemma played on fixed network g, thereafter called repeated network (Prisoner's Dilemma) game, at each point in discrete time, $t = 1, 2, ...$, the Prisoner's Dilemma stage game G is played.

Let player i's set of strategies be $F_i = \{\{f_i^t\}_{t=1}^\infty \mid f_i^1 \in A_i$, and for all $t > 1$, $f_i^t : \mathcal{H}_i^{t-1} \to A_i\}$. At any $t \geq 1$, player i's strategy $f_i = \{f_i^t\}_{t=1}^\infty$ prescribes him to choose an action. For $t > 1$, it maps his set of private histories to his action set. Let $F = \times_{i \in I} F_i$ be the repeated network game's strategy space and let strategy profile $f = (f_1, ..., f_n)$ be an element of F. Let player i's set of communication strategies be $Com_i = \{\{com_i^t\}_{t=1}^\infty \mid com_i^1 = \emptyset$, and for all $t > 1$, $com_i^t : \mathcal{H}_i^{t-1} \to R_i^t\}$. Player i's communication strategy $com_i = \{com_i^t\}_{t=1}^\infty$ prescribes him to send a report to all his neighbors at any $t > 1$. It maps his set of private histories to his report set. Let $Com = \times_{i \in I} Com_i$ be the repeated network game's communication space and let communication profile $com = (com_1, ..., com_n)$ be an element of Com. To emphasize player i's role, f is written as (f_i, f_{-i}) and com as (com_i, com_{-i}).

Let $F \times Com$ be the strategy and communication space of the repeated network game. At any $t \geq 1$, each pair $(f, com) \in F \times Com$ recursively generates a pure action profile $a^t(f, com) = (a_1^t(f, com), ..., a_n^t(f, com))$, a report profile $r^t(f, com) = (r_1^t(f, com), ..., r_n^t(f, com))$ and a corresponding observation profile $ob^t(f, com) = (ob_1^t(f, com), ..., ob_n^t(f, com))$. These determine the action and report profiles at $t + 1$.

If truthtelling is assumed or arises endogenously under strategic communication, player i only selects a strategy while his communication is determined as follows: at any $t > 1$, $com_i^t : (\mathcal{H}_i^{t-2}) \cup Ob_i^{t-1} \to R_i^t$ is such that the mapping from Ob_i^{t-1} to R_i^t is the identity. The players then hand over the repeated network game's true history.

Given a common discount factor $\delta \in [0, 1)$, the function $H^\delta : F \times Com \to \mathbb{R}^n$ assigns a payoff vector to each strategy and communication profile of the repeated network game. Given $(f, com) \in F \times Com$, player i's payoff, $H_i^\delta(f, com) = (1-\delta) \sum_{t=1}^\infty \delta^{t-1} h_i(a^t(f, com))$, is the $(1 - \delta)$-normalized discounted sum of stage game payoffs. Given δ and g, the repeated network Prisoner's Dilemma is defined as the normal form game $G^{g,\delta} \equiv (I, (F_i)_{i \in I}, (Com_i)_{i \in I}, (H_i^\delta)_{i \in I})$.

The players commonly know the game played, the network and the strategy choices available to all players, and importantly, observe their payoff only at the end of the

game.[4] The discount factor is then interpreted as the probability with which the Prisoner's Dilemma is played again in the next period. The probability that the repeated network Prisoner's Dilemma ended by period T converges to 1 as T goes to infinity.

2.4 Individual Rationality and Belief-free Equilibrium

A player's individually rational payoff is the lowest to which he can be forced in a stage game. It obtains when he maximizes his payoff while all other players minimize it and is called *minmax* payoff. For any player $i \in I$, let his minmax payoff in pure actions be

$$\nu_i \equiv \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} h_i(a_i, a_{-i}). \tag{1}$$

In the Prisoner's Dilemma, for any player the individually rational payoff is 1. It obtains when all players choose D. This is the unique stage game Nash Equilibrium in pure and strictly dominant actions. Denote by $\nu = (1, ..., 1)$ the minmax payoff vector.

The *set of feasible payoff vectors* of the repeated network game is defined as

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid \exists \{a^t\}_{t=1}^{\infty} : \forall t \geq 1,\ a^t \in A,\ \text{and}\ \forall i \in I,\ x_i = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} h_i(a^t)\}.$$

Any feasible payoff vector can be generated by a sequence of pure action profiles.[5]

The *set of feasible and individually rational payoff vectors* is denoted by $\mathcal{F}^*$. It contains all feasible payoff vectors that are larger than or equal to ν, and is defined as

$$\mathcal{F}^* = \{x \in \mathcal{F} \mid x \geq \nu\}.$$

Any payoff vector in this set is a candidate to be supported by a belief-free equilibrium.

In a belief-free equilibrium, each player conditions his action and report choices only on his observations. His strategy is a best-reply to the other players' strategies given his private history but not the other players' private histories. For a formal definition of BFE see Ely, Hörner and Olszewski (2005). In contrast to them, belief-freeness in this model arises not because the players are indifferent between choosing C and D, but rather since their action and report choices are only conditional on their observations. Moreover, these observations are precise and not only made with probability $(1 - \varepsilon)$.

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[4] If a player observes his or all players' payoffs before, then this could reveal information about other players' behavior which he did not (yet) observe via the network. The payoff would then be a (possibly imperfect) private or public signal. Both kinds of signal have been studied in the repeated games literature (see footnote 1), and to extend the results obtained here in this way is left for future research.

[5] Any payoff vector in $co(G)$ is feasible for $\delta \in (1 - \frac{1}{z}, 1)$, where z is the number of vertices of $co(G)$. For any discount factor in this range, sets $\mathcal{F}$ and $co(G)$ coincide; see Fudenberg, Levine and Maskin (1994).

Definition 1. *Strategy profiles* $(\dot{f}, \dot{com}) \in F \times Com$ *are a* belief-free equilibrium (BFE) *of* $G^{g,\delta}$, *if for all* $t \geq 1$ *and given any* $\cup_{s=1}^{t} ob^{s}$, $\{\dot{f}^{\tau}, \dot{com}^{\tau}\}_{\tau=t+1}^{\infty}$ *are such that for all* $i \in I$, *and all* $(f_i, com_i) \in F_i \times Com_i$,

$$(1-\delta) \sum_{s=t+1}^{\infty} \delta^{s-1} h_i(a^s(\dot{f}, \dot{com}) \geq (1-\delta) \sum_{s=t+1}^{\infty} \delta^{s-1} h_i(a^s(f_i, \dot{f}_{-i}, com_i, \dot{com}_{-i})).$$

The set of BFE strategy profiles is denoted by $BFE(G^{g,\delta})$. If truthtelling is imposed exogenously $BFE^{ET}(G^{g,\delta})$ is adored with superscript ET for *exogenous truthtelling*. A strategy profile is a BFE if, and only if, no player's finite unilateral deviation at any point in time is profitable.[6]

3   The Network makes a difference

The following example of the Prisoner's Dilemma with three players and exogenously imposed truthtelling, denoted by $\hat{com} \in Com$, illustrates how imposing a network on a set of players affects the set of BFE. Consider the trigger strategy profile. It prescribes each player to cooperate as long as all players cooperate and to defect forever if any player defected. Given any network g, the trigger strategy of player i, denoted by $\hat{f}_i \in F_i$, is defined as follows: $\hat{f}_i^1 = C$, and for $t \geq 1$, given $ob_i^t \in Ob_i^t$,

$$\hat{f}_i^{t+1}(ob_i^t) = \left\{ \begin{array}{ll} D & \text{if } \exists\ 1 \leq \tau \leq t \text{ such that for } a^\tau \in ob_i^t,\ a_j^\tau = D,\ \text{while } a_{-j}^\tau = C \\ C & \text{otherwise.} \end{array} \right.$$

Given $(\hat{f}, \hat{com}) \in F \times Com$, for all $i \in I$ and all $t \geq 1$, first $a_i^t(\hat{f}, \hat{com}) = C$, and second, for all $a_j^\tau \in ob_i^t(\hat{f}, \hat{com})$, $a_j^\tau = C$ as well for all $1 \leq \tau \leq t$ and all $j \in I$. Hence, for all $i \in I$, $H_i^\delta(\hat{f}, \hat{com}) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} h_i(a^t(\hat{f}, \hat{com})) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} 3 = 3$.

3.1   The Players form a Star

Consider a *star* (or a line) with $n = 3$, as represented in Figure 1. The graph of g is $E = ((1,2),(2,1),(2,3),(3,2))$. Figure 2 represents G for $n = 3$, where player 1 chooses rows, player 2 columns and player 3 matrices. The trigger strategy profile is a BFE of $G^{g,\delta}$ under truthtelling if, and only if, all players are patient enough, that is, δ is higher than some threshold value. Then, none of them ever deviates. Corresponding conditions on δ must hold for the truncation of the repeated network Prisoner's Dilemma at any

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[6]Since $\delta < 1$, a player's gain from a deviation of infinite length can be approximated by that of a finite deviation. Therefore, unilateral deviations of finite length from a strategy profile are not profitable if, and only if, it is a BFE of the repeated network game; see Mailath and Samuelson (2006).

$$1\text{---}2\text{---}3$$

Figure 1: Three players form a Star

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C   D

1-2	C	D
C	3, 3, 3	0, 4, 0
D	4, 0, 0	2, 2, 0

1-2	C	D
C	0, 0, 4	0, 2, 2
D	2, 0, 2	1, 1, 1

Figure 2: Prisoner's Dilemma for three players

point in time, that is, given any observation profile. A BFE does not impose restrictions on play after a multilateral deviation by two or more players. Any unilateral deviation that may arise can be uniquely allocated to one of the following three classes:

1) initial unilateral deviations,

2) subsequent unilateral deviations (before the initial is known by all players), and

3) unilateral deviations when the punishment takes place.

Obviously, unilateral deviations during the punishment are not profitable since all players choose D. The resulting action profile is the stage game Nash Equilibrium in strictly dominant actions. Hence, every player best-replies independently of g and of δ. For the same reason, no player can deviate profitably from the trigger strategy profile in class 2. After a player's initial deviation, he and any player who knows about it are best-off to play D forever (rather than to deviate and to choose C at any point in time).

It remains to show that no player has a profitable unilateral deviation from the trigger strategy profile when all players choose C. Given δ, player 2 (who is directly observed by 1 and 3) does not deviate in any period τ if, and only if,

$$(1-\delta)\sum_{t=1}^{\infty} 3\delta^{t-1} \geq (1-\delta)\sum_{t=1}^{\tau-1} 3\delta^{t-1} + 4(1-\delta)\delta^{\tau-1} + (1-\delta)\sum_{t=\tau+1}^{\infty} 1\delta^{t-1},$$

$$(1-\delta)\sum_{t=\tau+1}^{\infty} 2\delta^{t-1} \geq (1-\delta)\delta^{\tau-1},$$

$$2\delta^{\tau+1} \geq (1-\delta)\delta^{\tau},$$

$$\delta \geq \tfrac{1}{3}.$$

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The value of $\frac{1}{3}$ is not only the threshold value for player 2 in this example but also the one for all players in a complete network. The network affects, however, the threshold value of the remaining two players in this example. Given δ, player 1 (and similarly 3) does not deviate from the trigger strategy profile in any period τ if, and only if,

$$(1-\delta)\sum_{t=1}^{\infty}3\delta^{t-1} \geq (1-\delta)\sum_{t=1}^{\tau-1}3\delta^{t-1}+4(1-\delta)\delta^{\tau-1}+2(1-\delta)\delta^{\tau}+(1-\delta)\sum_{t=\tau+2}^{\infty}1\delta^{t-1},$$

$$(1-\delta)\delta^{\tau}+(1-\delta)\sum_{t=\tau+2}^{\infty}2\delta^{t-1} \geq (1-\delta)\delta^{\tau-1},$$

which simplifies to $2\delta+\delta^2-1\geq 0$. The only positive solution for δ in this quadratic equation is $\delta \approx 0.414$. Hence, in class 1 of the BFE conditions the requirement on δ, or the players' patience, is higher in the three players star network than in a complete one due to the one period lag with which players 1 and 3 observe each other's action choice.

This example extends to any star network where $n > 3$. The player at the center of the star has the same role as player 2 in this example, and for all other players the same conditions apply as for players 1 and 3 in this example.

3.2 The Repeated Prisoner's Dilemma Played in any Network

A similar result holds for any network (with $n > 3$), in which all players follow the trigger strategy. In Figure 1 it takes 2 periods until full punishment sets in. In any network, it takes d_i periods until all players punish player i. The group of punishers increases strictly until d_i periods after i's deviation. Until then the deviator's payoff is 2 since at least one player still chooses C. Thereafter, it is 1 forever.

Since the diameter d is the maximal largest distance among all players, for any network g, there is a discount factor δ^* that solves $2\delta^*+\delta^{*d}-1\geq 0$ such that no player in the network deviates from the trigger strategy profile. Hence, for this strategy profile it is possible to classify all networks that can be formed from the set of players according to their diameter. The threshold value of the discount factor δ^*, for which no player deviates from the trigger strategy profile, that is, the level of patience required to sustain cooperation is non-decreasing in the network's diameter d. Intuitively, a higher diameter implies that information between at least one pair of players travels over a longer distance.

Although the expression $2\delta^*+\delta^{*d}-1\geq 0$ depends on d, even in large networks the threshold value for δ^* is bounded above by $\frac{1}{2}$. To see this, take the limit of the inequality when d converges to infinity. Since $\delta < 1$, the term δ^{*d} converges to 0 and the inequality

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simplifies to $\delta^* \geq \frac{1}{2}$. Hence, for "moderately patient" players, the trigger strategy profile is a BFE in any repeated network Prisoner's Dilemma.

3.3 The Network is Unconnected

Network g is unconnected if there are different connected and undirected components. Suppose that each component contains at least two players. Together all components constitute network g. All players in a component observe each other and communicate with each other. They never observe any player in another component nor do they receive reports about their action choices—the distance between any pair of unconnected players is normalized to infinity. However, all players (in the different components) still participate in a single Prisoner's Dilemma game at every point in time.

Suppose that all players in the distinct components follow a modified trigger strategy profile. Any player's unilateral deviation is only punished by the players in the component, since no other player ever observes it, although it affects any other player's payoff. The players (in his component) who observe the deviation choose D forever (possibly after some delay). This is a BFE if, and only if,

$$(1-\delta)\sum_{t=1}^{\infty} 3\delta^{t-1} \geq (1-\delta)\sum_{t=1}^{\tau-1} 3\delta^{t-1} + 4(1-\delta)\delta^{\tau-1} + (1-\delta)\sum_{t=\tau+1}^{\infty} 2\delta^{t-1},$$

$$(1-\delta)\sum_{t=\tau+1}^{\infty} \delta^{t-1} \geq (1-\delta)\delta^{\tau-1},$$

$$\delta^{\tau+1} \geq (1-\delta)\delta^{\tau},$$

$$\delta \geq \tfrac{1}{2}.$$

This result shows that for moderately patient players cooperation is a BFE even if a player never observes the actions chosen by some other participants of the game. It also holds under strategic communication as follows from Theorem 1 in section 4 and even if the number of players becomes arbitrarily large. Although the value of $\frac{1}{2}$ crucially depends on the parameter choice in the Prisoner's Dilemma in Figure 2, identical qualitative results would arise in a parameterized Prisoner's Dilemma game.

4 Strategic Communication

In this section, the Prisoner's Dilemma played on any network is extended to strategic communication. This is challenging due to the bilateral communication structure. To illustrate some of the encountered difficulties, first an example is provided. Then, the trigger strategy and truthtelling are shown to be a BFE under strategic communication. From this two corollaries follow. Finally, a BFE with richer than truthful communication is derived and a Folk Theorem is established.

Given any observation profile, unilateral deviations from the strategy, from the communication and from both have to be shown to be unprofitable. In particular, deviations from the communication profile under truthtelling, that is, lies have to be dealt with. Two kinds of lies may occur. A player claims that there was a deviation when there was none or he does not reveal a deviation and neither punishes it.

Example 1 illustrates the situation in which lying is most difficult to prevent since one player, called monitored player, has only one monitor.

Example 1. *Suppose that the players are asked to tell the truth and to punish any deviation from the strategy or communication. In this case, the monitor, after observing a deviation of the monitored player is asked to report and to punish it. Does he have an incentive to lie, by not revealing it, and to deviate simultaneously by not punishing it?*

Suppose that the monitor lies and continues to follow the sequence of action profiles as if the deviation had not occurred. Then, the monitor is the last deviator and the monitored player, observing the monitor's deviation, starts to punish him. Thus, in equilibrium, the monitor is better off to start punishment of the monitored player and to report the deviation truthfully. In case the monitor and the monitored player deviated together, and instead of starting punishment continue to play the initially prescribed sequence of action profiles, then this is a multilateral deviation which is ignored in a BFE. Henceforth, in a BFE, the monitor reports the monitored player's deviation truthfully.

The trigger strategy and truthtelling are a BFE under strategic communication as is shown in Theorem 1. In this case, a deviation from the communication profile is a lie.

Theorem 1. *Let G, g, $(\hat{f}, c\hat{o}m) \in F \times Com$ and $\delta \in [0,1)$ be given. Then, $(\hat{f}, c\hat{o}m) \in BFE^{ET}(G^{g,\delta})$ if, and only, if $(\hat{f}, c\hat{o}m) \in BFE(G^{g,\delta})$.*

Proof. Suppose that $(\hat{f}, c\hat{o}m) \in BFE^{ET}(G^{g,\delta})$. Then, unilaterally choosing a different action than prescribed by $\hat{f}$ after any history, even if it includes lies, is not profitable for any player since $(\hat{f}, c\hat{o}m) \in BFE^{ET}(G^{g,\delta})$.

Next it is shown that a lie and deviation are unprofitable given any observation profile. Three cases might occur. First, no player has yet chosen D. If a player nevertheless claims that some other chose D, he starts punishment and chooses D. He lies and deviates since $\hat{f}$ prescribes him to choose C. Punishment starts as if the player who lied had deviated himself. Since he cannot deviate profitably, he neither can lie and deviate profitably.

Suppose next that at least one player has chosen D already. Any player who observed this should choose D to punish the deviator. If a player lies and claims that there was no deviation and deviates by choosing C instead of D, as prescribed by $\hat{f}$, he is worse off since at least the deviator chooses D by $\hat{f}$. Finally, suppose that all players choose D. Then, all of them are indifferent to tell the truth or not since to claim that any player chose C instead of D, even if this were true, does not change the sequence of action profiles played. To lie and to choose C, in this case, is neither profitable.

Finally, no player's lie is profitable given any observation profile. A player is indifferent to reveal a deviation from C to D. If he does not reveal it, but still punishes it by choosing D he starts punishment anyway. Similarly, if he observes a deviation from D to C, he is indifferent to reveal it since anyway punishment started and he continues to choose D. Hence, lies (without deviation from the strategy) do neither occur after any history which already includes a sequence of deviations and/or lies.

Suppose that $(\hat{f}, c\hat{o}m) \in BFE(G^{g,\delta})$. Then, no player ever deviates from $\{a^t(\hat{f}, c\hat{o}m), r^t(\hat{f}, c\hat{o}m)\}_{t=1}^{\infty}$, and neither if truthtelling is imposed exogenously. $\qquad\square$

If the players in any network follow the trigger strategy profile and $\delta \geq \frac{1}{2}$, as shown in section 3.2, no player ever deviates from the strategy or communication profile or from both, and truthtelling arises endogenously. For some networks, this result holds even for values of $\delta \in (\frac{1}{3}, \frac{1}{2})$.

By the threat of trigger punishment, it is possible to sustain other sequences of action profiles under strategic communication as BFE. Let $\bar{f}$ denote the strategy profile which prescribes for any sequence of action profiles $\{a^t\}_{t=1}^{\infty}$ a conversion to D forever after observing any inconsistency in the strategy or communication. Given $\bar{f}$ and $c\hat{o}m$, if δ is large enough, no player ever deviates or lies, that is, truthtelling arises endogenously under strategic communication and $(\bar{f}, c\hat{o}m)$ is a BFE. This is stated formally in Corollary 1.

Corollary 1. *Let G, g, $x \in \mathcal{F}^*$ and $(\bar{f}, c\hat{o}m) \in F \times Com$ be given. Then, there is $\bar{\delta} \in [0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$, there is $\{\bar{a}^t\}_{t=1}^{\infty}$ such that $\{a^t(\bar{f})\}_{t=1}^{\infty} \equiv \{\bar{a}^t\}_{t=1}^{\infty}$, $x_i = H_i^{\delta}(\bar{f}, c\hat{o}m)$ for all $i \in I$, and $(\bar{f}, c\hat{o}m) \in BFE^{ET}(G^{g,\delta})$ if, and only if, $(\bar{f}, c\hat{o}m) \in BFE(G^{g,\delta})$.*

The proof of this corollary is straightforward. It combines arguments from the proof of Theorem 1 with mathematical calculations analogous to that in section 3.

That, after observing a deviation, all players choose D, at least for some time, is a powerful threat. Partly, since it is not required to know the deviator's name but only that a deviation occurred. As long as a player identifies a deviation he can punish it and another corollary follows immediately.

Corollary 2. *Let G, g, $(f, c\hat{o}m) \in F \times Com$ and $\delta \in [0, 1)$ be given. Then, $(f, c\hat{o}m) \in BFE^{ET}(G^{g,\delta})$ if, and only if, $(f, c\hat{o}m) \in BFE(G^{g,\delta})$.*

Corollary 2's proof is analogous to that of Theorem 1: If $(f, c\hat{o}m)$ are a BFE under exogenously imposed truthtelling, then unilateral deviations are neither profitable under strategic communication.

However, under strategic communication some player's lie may be part of a BFE, and thus the set of BFE under strategic communication is not a subset of that under truthtelling. Suppose that a player is prescribed not to report certain observations which would trigger punishment by some players. Any network in which one player has only one monitor is prone to this kind of equilibrium, as is shown in Theorem 2.

Theorem 2. *Let G, $\delta \in (\frac{2}{3}, 1)$ and g be given such that one player in g has only one monitor. Then, there is $(f, com) \in BFE(G^{g,\delta})$ such that $(f, com) \notin BFE^{ET}(G^{g,\delta})$.*

Proof. Let g be such that one player, called monitored player, is only connected to his monitor. Let f be a modified trigger strategy profile: each player chooses C at any t, and D forever after observing any player having unilaterally chosen D or after observing any inconsistent report. The monitor and the monitored player's strategy is identical, except of the monitored player who is prescribed to choose D at $t = 1000$, and both, the monitor and the monitored player, ignore this observation of D and continue with the initially prescribed sequence of action profiles.

Moreover, *com* prescribes all players to report truthfully any observation they made, except of the monitor. He reports the monitored player's action choice in any other period than 1000 truthfully. At $t = 1001$, he reports that the monitored player chose C in the previous period. The monitor and the monitored player do not punish this lie, though they punish any other lie they identify. All other players punish any lie they observe by converting to D forever.

Then, for $\delta \in (\frac{2}{3}, 1)$, as is easily verified, $(f, com) \in BFE(G^{g,\delta})$, but $(f, com) \notin BFE^{ET}(G^{g,\delta})$ since under *com* the monitor is prescribed to lie. □

Even under exogenously imposed truthtelling, $(f, c\hat{o}m) \notin BFE^{ET}(G^{g,\delta})$. Given $(f, c\hat{o}m)$, the monitor truthfully reports the monitored player's choice in every period.

The other players anticipate the monitored player's choice of D in period 1000 and it is commonly known that cooperation breaks down in this period. As in any finitely repeated Prisoner's Dilemma with perfect monitoring, cooperation then is unsustainable from the beginning on.

An interesting feature of this result is that the players prefer to be lied to, rather than to receive the unpleasant truth of the monitored player's choice of D. Although it is common knowledge that the monitored player chooses D in period 1000,[7] all players except of the monitor and the monitored player never observe this choice of D and are better off to follow their strategy which is not conditioned on the common knowledge of the monitored player's choice of D. In equilibrium, a player can permit himself to stand on a high moral ground and threatens to punish any player's choice of D since he is sure never to receive an observation after which he would have to carry out his threat. In case the monitor were to communicate his knowledge of the monitored player's choice of D throughout the network, first this would not be a BFE, and second, the players in this case would want to jointly adopt the monitor and the monitored player's strategy of not punishing it.

The following corollary states formally that under strategic communication additional BFE arise compared with exogenously imposed truthtelling.

Corollary 3. *Let G, g and $\delta \in [0, 1)$ be given. Then, $\{(f, com) \in F \times Com \mid (f, com) \in BFE^{ET}(G^{g,\delta})\} \subseteq \{(f, com) \in F \times Com \mid (f, com) \in BFE(G^{g,\delta})\}$.*

Thus, every BFE under exogenously imposed truthtelling is sustainable under strategic communication, though there are BFE under strategic communication which are not compatible with truthtelling. This follows trivially from Corollary 2 and Theorem 2.

Finally, it is possible to establish a Folk Theorem for the repeated network Prisoner's Dilemma under strategic communication, that is, every feasible and strictly individually rational payoff vector can be supported by a BFE strategy and communication profile if the players are patient enough. This corollary is a consequence of Corollaries 1 and 2.

Corollary 4. *Let G and g be given. Then, for all $x \in \mathcal{F}^*$, there is $\tilde{\delta} < 1$ such that for each $\delta \in (\tilde{\delta}, 1)$, there are $(\tilde{f}, \tilde{com}) \in F \times Com$ such that $(\tilde{f}, \tilde{com}) \in BFE(G^{g,\delta})$ and $H^\delta(\tilde{f}, \tilde{com}) = x$.*

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[7] This is a consequence of the common knowledge of the game and strategy profile.

5 Final Remarks

5.1 Related Literature

In the repeated games literature, there are different approaches to model imperfect private monitoring in an infinitely repeated Prisoner's Dilemma. Usually, the imperfection of the monitoring technology is imposed exogenously and limit results are given when it vanishes. Several papers in a special edition of the Journal of Economic Theory in 2002 provide corresponding results. Bhaskar and Obara (2002) and Sekiguchi (1997) analyze belief-based sequential equilibria, while Ely and Välimäki (2002) and Piccione (2002) study BFE in which the players that participate in an infinitely repeated Prisoner's Dilemma with imperfect private monitoring are indifferent between choosing C and D at any point in time.

The model studied here is using a simpler version of BFE. A player need not form beliefs about the history of his opponents since his observations are precise and not only made with probability $(1 - \varepsilon)$. Belief-freeness arises since the players condition their strategy only on their observations, while in Ely, Hörner and Olszewski (2005) and in the above mentioned papers the players have to be indifferent among different actions in order to achieve it. Imperfect private monitoring in the repeated network Prisoner's Dilemma is not caused by an exogenously imposed monitoring technology, but rather by the players' strategic decisions.

Strategic communication in networks can be modelled in various ways.[8] The approach taken in this paper also relates to the literature on communication in repeated games.[9]

Few papers combine both ideas. Ben-Porath and Kahneman (1996) study sequential equilibria of infinitely repeated discounted games in which the players form a (not necessarily connected) network. The players publicly announce their own action choices and observations made about their neighbors in a strategic way, that is, including lies. When each group contains three or more players unilateral deviations are detectable, and hence, do not occur in equilibrium. In Ben-Porath and Kahneman (2003) monitoring, moreover, is costly. Thus, only one monitor is assigned to every player. After an incompatible announcement, which in equilibrium does not occur, both players are punished and the monitor is substituted. In comparison to both papers, the network in this paper is connected, though as seen in section 3.3, the trigger strategy is a BFE also in unconnected networks, as long as each component of the network contains at least two players.

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[8] See Hagenbach and Koessler (2009) for one possibility and further references.

[9] See for example Compte (1998), Kandori and Matsushima (1998) or Kandori (2003), who all resolve imperfect monitoring in repeated games by communication in form of public announcements.

5.2 The most central or best informed Player

There are two ways to identify the most central or best informed player. Depending on the communication profile one or both determine this player's location.

Under exogenously imposed truthtelling, the most central player is the one whose largest distance is smallest. He is first informed about all other players' action choices at some point in time. The second concept is Bonacich centrality, as defined by Ballester, Calvó-Armengol and Zenou (2006). Roughly, it counts the number of paths of different length which start in any player $i \in I$, weighted by the discount factor δ. The player from which more paths stem is most central. Under strategic communication, he receives the most information which includes what his neighbors tell him that he told them that they told him and so on. His informational advantage might be of quantitative rather than qualitative nature. Moreover, the other players in the network accumulate more information about him than about any other player.

Under truthtelling only largest distances matter while under strategic communication both concepts are important. Bonacich centrality identifies the player with more information and largest distances the one who first receives (possibly wrong) information about all other players' action choices. Ballester, Calvó-Armengol and Zenou (2006) show that both concepts do not coincide, and usually identify different players as being most central.

5.3 Conclusion

Although the Prisoner's Dilemma is a well-studied game, there are still new results to explore. This paper studies the imposition of a communication network on a set of impatient players in the repeated Prisoner's Dilemma. The players' level of patience required to sustain the trigger strategy profile as a BFE is larger even in a simple three players star network compared with a complete one. For sufficiently patient players, the trigger strategy profile is a BFE in unconnected networks and under strategic communication. Any BFE under exogenously imposed truthtelling is also a BFE under strategic communication when the players are prescribed to tell the truth. New BFE arise due to richer communication and the set of BFE under strategic communication is not a subset of that under exogenously imposed truthtelling.

That some player's lie—from which all players benefit—is part of a BFE, is a feature frequently observed in reality. In many societies or groups, the existence of "misdeeds" or "skeletons in the cupboard" is well-known, though no action is taken as long as these are not observed publicly. The agent who withholds corresponding information has the

same role as the monitor in Theorem 2.

The results obtained in this paper readily extend to directed networks and to observation and communication structures in which information does not flow one link per period, for example, since every player takes a different amount of time to process information.

The model is not presented in mixed actions since a player cannot be forced to a lower *minmax* payoff using mixed actions, and since all feasible and strictly individually rational payoff vectors can be generated by sequences of pure action profiles.

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